2025 CMWMC Relay Round Solutions

1.1. Define $\gcd(a, b)$ to be the largest common divisor of integers a and b. Likewise, define $\text{lcm}(a, b)$ to be the least common multiple of integers a and b. Suppose x,y are positive integers such that $x + y = 45$. Compute the maximum value of $\text{lcm}(x, y) + \gcd(x, y)$.

Proposed by Michael Duncan

Answer. 507

Solution. The maximum possible value of the lcm of two numbers would be their product, which happens when the two numbers are coprime. The maximum possible product of two numbers summing to 45 is $22 \cdot 23 = 506$ (when the numbers are closest together), which is also the lcm since 22 and 23 are coprime. This is the best we can do, so

$$\text{lcm}(22, 23) + \gcd(22, 23) = 506 + 1 = \boxed{507}$$

1.2. Let T denote the answer to the previous problem. Let N be the remainder when T is divided by 100. Compute the number of ordered pairs of **non-negative integers** (x, y) such that

$$20x + 25y = 100N$$

Proposed by Lohith Tummala

Answer. 8

Solution. We simplify to get $4x + 5y = 20N$. If we set $y = 0$, we get $x = 5N$. If we set $x = 0$, we get $y = 4N$. These are the extremes of the solution set; any other solutions satisfy $x \in [0, 5N]$ and $y \in [0, 4N]$. Note that from (x, y), we can get another integer solution by increasing x by 5 and decreasing y by 4. This works since if $4x + 5y = 20N$,

$$4(x + 5) + 5(y - 4) = 4x + 5y + 20 - 20 = 20N$$

Thus, the solution set is

$$(0, 4N), (5, 4(N - 1)), (10, 4(N - 2)), \cdots , (5N, 0)$$

This gives us $N + 1$ total solutions. Here, $N = 7$, so the answer is $\boxed{8}$.

1.3. A positive integer x in base 10 has three digits. When x is expressed in base T, it also has three digits. When x is expressed in base $2T$, it yet again has three digits. How many possible values of x are there?

Proposed by Lohith Tummala

Answer. 256

Solution. Having three digits in base 10 implies that $100 \leq x \leq 999$.
Having three digits in base T implies that $T^2 \leq x \leq T^3 - 1$.
Having three digits in base $2T$ implies that $(2T)^2 \leq x \leq (2T)^3 - 1$. Before getting T, we can simplify the bottom two inequalities to $(2T)^2 \leq x \leq T^3 - 1$. Given that $T = 8$, we have

$$100 \leq x \leq 999$$

$$256 \leq x \leq 511$$

Turns out the first inequality is useless. There are $511 - 256 + 1 = \boxed{256}$ possible values of x.

2.1. Henry has a weight scale that is slightly broken. When he places n fruits on the scale, it calculates the total weight of the fruits, but adds on an amount proportional to the number of fruits before reporting. Specifically, it will add nk, where k is always constant and unknown to Henry.

Henry places 2 apples and 3 bananas on the scale and the scale measures 60 g. He then places 4 apples and 5 bananas and the scale measures 110 g. What will the scale report when 5 apples and 5 bananas are placed, in grams? (All apples are identical and all bananas are identical).

Proposed by Lohith Tummala

Answer. 125

Solution. Let's make equations for this. If a is the number of apples and b is the number of bananas, we have

$$2a + 3b + 5k = 60$$

$$4a + 5b + 9k = 110$$

If we subtract the first equation from the second, we get

$$2a + 2b + 4k = 50$$

$$\implies a + b + 2k = 25 \implies 5a + 5b + 10k = \boxed{125}$$

2.2. Krish is walking up an escalator. The escalator is partially functional; it will start by moving 2 stairs per second for five seconds, then stop moving for five seconds, and this cycle continues. Krish walks at a constant rate of 3 stairs per second. How many seconds will it take Krish to climb a T-stair escalator if he is always walking?

Proposed by Lohith Tummala

Answer. 31

Solution. One pitfall to avoid is that even though the escalator has T stairs, Krish does not necessarily walk T stairs. When the escalator is moving, Krish clears the distance of $d = rt = (2 + 3)5 = 25$ stairs in five seconds. When the escalator is not moving, Krish clears the distance of $d = rt = (3)5 = 15$ in five seconds. Thus, every five seconds, Krish clears the distances $25, 15, 25, 15, \cdots$. Given that $t = 125$, we clear the distances $25 + 15 + 25 + 15 + 25 + 15 = 120$ in 30 seconds, with five more stairs left. The escalator turns on at this moment, so it takes Krish one more second to clear the distance of five stairs, giving a total of 31 seconds.

2.3. Kyle draws the circle $x^2 + (y - T)^2 = 16$ and the parabola $y = x^2 + k$. Find the **minimum** value of k such that these graphs intersect.

Proposed by Kyle Hynes

Answer. 59/4 (14.75)

Solution. Plug in $x^2 = y - k$ into the circle equation and solve:

$$y - k + (y - T)^2 = 16$$
$$y^2 + y(-2T + 1) + T^2 - k - 16$$
$$y = \frac{2T - 1 \pm \sqrt{(1 - 2T)^2 - 4(T^2 - k - 16)}}{2}$$

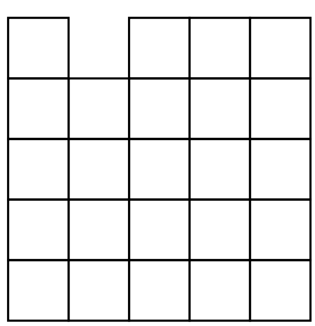
We only need to look at the discriminant D. If $D < 0$, we have no intersection points. Thus, $D \geq 0$.

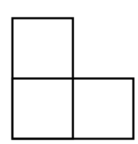
$$(1 - 2T)^2 - 4(T^2 - k - 16) \geq 0$$
$$1 - 4T + 4T^2 - 4T^2 + 4k + 64 \geq 0$$
$$65 - 4T + 4k \geq 0$$
$$4k \geq 4T - 65$$
$$k \geq T - \frac{65}{4}$$

Since $T = 31$, we have the minimum value of k being

$$k = 31 - \frac{65}{4} = \boxed{\frac{59}{4}}$$

3.1. Colin is tiling the following 5×5 grid with the second square removed. He begins placing L shaped triomino tiles (shown below to the right) in any orientation such that each tile covers exactly 3 squares of the grid and there is no overlap among any tiles. What is the maximum number of tiles he can place?





L-shaped triomino

Proposed by Alan Abraham

Answer. 7

Solution. With some trial and error, you can find a lot of solutions with 7 tiles, and you may be wondering if 8 tiles is possible. The short answer is no. The long answer is the following. If it was possible, we would have to cover every square since there are 24 squares.

21		22	23	24
16	17	18	19	20
11	12	13	14	15
6	7	8	9	10
1	2	3	4	5

So let's try to optimize as best we can. First, we have to place a triomino on $(21, 16, 17)$. Next, let's figure out square 1. If we place a triomino on $(1, 6, 7)$, this is impossible since nothing can be placed on 11. Thus, we can deduce that any triomino placed on 1 will force another triomino placement, and the two triominoes collectively cover $(1, 2, 6, 7, 11, 12)$.

We now have three triominoes placed, and have a 5×3 grid left. Let's try to place a triomino on 5. If we try $(9, 4, 5)$, this traps 3. The only way is to use two triominoes to cover $(8, 9, 10, 3, 4, 5)$, leaving us a 3×3 grid $(22, 23, 24, 18, 19, 20, 13, 14, 15)$ with three triominoes left.

At this point, if you place a triomino on 15, you either (a) force another triomino on $(13, 18, 19)$ and you're left with a row of three, (b) force another triomino on $(20, 23, 24)$ and you're left with a row of three, or (c) force another triomino on $(13, 14, 18)$ and you're left with a row of three. You'll always be stuck at $\boxed{7}$ triominoes placed.

3.2. Let T denote the answer to the previous problem. Let $K = T + 4$. Sam and Jenny are trying to decide who gets to eat the last Subway Cookie with a game of rock paper scissors. Sam's strategy is to pick rock with probability $\frac{2}{K}$, paper with probability $\frac{6}{K}$, and scissors with probability $\frac{K-8}{K}$. Hearing this, Jenny decides her strategy to independently pick rock with probability $\frac{K-8}{K}$, paper with probability $\frac{2}{K}$, and scissors with probability $\frac{6}{K}$.

The probability that Sam wins in one game is $\frac{N}{K^2}$. What is N?

Proposed by Alan Abraham

Answer. 36

Solution. There are only three choices for what Jenny picks, the probability Sam wins for each case is Sam's probability of picking the counter times Jenny's probability of picking the choice.

$$\frac{K-8}{K} \cdot \frac{6}{K} + \frac{2}{K} \cdot \frac{K-8}{K} + \frac{6}{K} \cdot \frac{2}{K} = ((K-8) \cdot 6 + 2 \cdot (K-8) + 6 \cdot 2)/K^2$$

We only want N, so we can just compute the numerator. $(K-8)\cdot 6+2\cdot(K-8)+6\cdot 2 = 6K-48+2K-16+12 = 8K-52 = 8(T+4)-52 = 8T+32-52 = 8T-20$. Now we can plug in $T=7$ to get $8\cdot 7-20 = 56-20 = \boxed{36}$.

3.3. Let T denote the answer to the previous problem. Michael is building four towers with his limitless supply of blocks. His towers are initially T^2, $T(T+1)$, $T(T+1)$, and $(T+1)^2$ blocks high, respectively. We consider a "*move*" to be choosing three distinct towers, and placing 1 block on top of each of them. What is the minimum amount of moves required in order for the heights of all the towers to be equal?

Proposed by Michael Duncan

Answer. 145

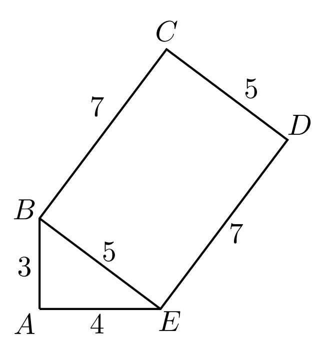
Solution. Notice that this operation is effectively equivalent to taking 1 block away from the fourth tower. Then we can reduce the 2nd, 3rd, and 4th towers to the height of the first tower. For the second and third towers, this takes $T(T+1)-T^2=T$ operations each. For the fourth tower, this takes $(T+1)^2-T^2=2T+1$. So our answer is $4T+1=4\cdot 36+1=\boxed{145}$.

4.1. Let convex pentagon $ABCDE$ have sides $AB=3, BC=7, CD=5, DE=7$, and $EA=4$. Furthermore, angles A, C, D are right angles. What is the area of this pentagon?

Proposed by Lohith Tummala

Answer. 41

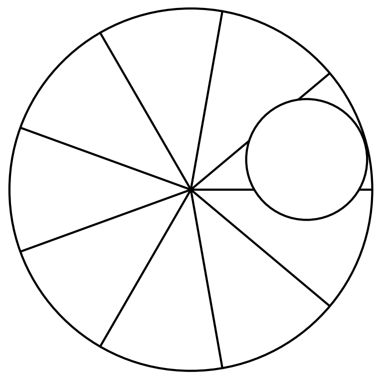
Solution. Let's draw the diagram and draw $\overline{BE}$.



As we can see, this is a right triangle and a rectangle. The area is

$$\frac{3\cdot 4}{2}+7\cdot 5=6+35=\boxed{41}$$

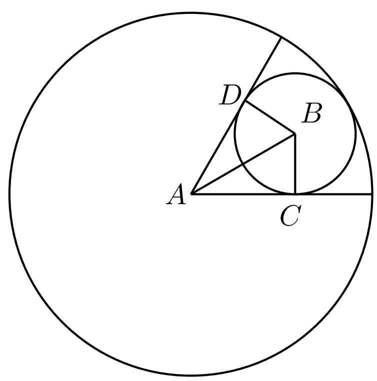
4.2. Let T denote the answer to the previous problem. On a piece of paper, Dustin draws a circle of radius 6 and draws lines to split the circle into T equal sectors. April has a circular sticker of radius 2 and places it on the drawing such that it is internally tangent to the bigger circle. After this, x sectors are **not** fully visible. Find the **minimum** possible value of x. (As an example, if $T = 9$, the below sticker placement leaves 3 regions not fully visible.)



Proposed by Lohith Tummala

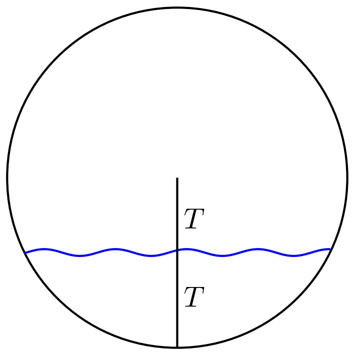
Answer. 7

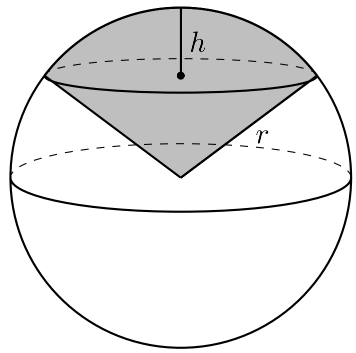
Solution. We'll draw the same diagram but with the sticker position adjusted:



Let O be the large circle with center A and P be the the small circle with center B. Draw the line AB, which has length 4. Draw a radius of O that is tangent to P at point C. Draw a radius of O that is tangent to P at point D on the opposite side of P from C. Now BC and BD have length 2. Now any sector of O that intersects P has one of its bounding radii within DAC. Let θ be the angle BAC. We have that $\sin(\theta) = BC/AB = 1/2 \implies \theta = 30°$. So P intersects any sector in some $60°$ region of O. Each sector is $360°/T$, so the combined angle of x sectors is $x \cdot 360°/T$. Now we know this combined angle $x \cdot 360°/T \geq 60° \implies 6x \geq T$. Since $T = 41$, the minimum value of x is $\boxed{7}$.

4.3. Let T denote the answer to the previous problem. A spherical yoga ball has radius $2T$ inches. Michael takes a hose and fills the yoga ball with some water. As a result, the height of the water in the ball is T inches (as illustrated on the left). Michael then adds Z more cubic inches of water to completely fill the yoga ball. Find Z. (Note: A spherical sector, as pictured on the right, has a volume $V = \dfrac{2\pi r^2 h}{3}$, where r is the radius of the sphere and h is the height of the spherical "cap".)





Proposed by Lohith Tummala

Answer. 3087π

Solution. We want the volume of the water, which forms the shape of a spherical cap. The volume of the spherical cap is equal to the volume of the spherical sector minus the volume of the cone in that sector. Doing this, we get

$$\frac{2\pi(2T)^2(T)}{3} - \frac{\pi(T\sqrt{3})^2(T)}{3} = \frac{\pi}{3}(8T^3 - 3T^3) = \frac{5\pi T^3}{3}$$

(Note the radius of the cone. This is because of the $30-60-90$ triangle between the cone radius, cone slant height of length $2T$, and cone height of length T.)

We now need to subtract this from the sphere's volume to get the remaining volume of water needed.

$$\frac{4\pi(2T)^3}{3} - \frac{5\pi T^3}{3} = \frac{\pi T^3}{3}(32-5) = \frac{27\pi T^3}{3} = 9\pi T^3$$

Since $T = 7$, we have a remaining volume of $\boxed{3087\pi}$.