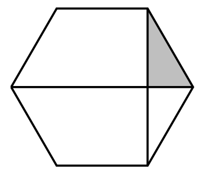
2025 CMWMC Individual Round Solutions

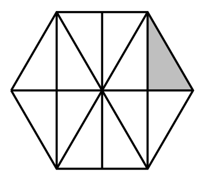
1. The area of the shaded region is 5. What is the area of the entire hexagon, given that it is regular?



Proposed by Lohith Tummala

Answer. 60

Solution. If we tile as follows, we get that the hexagon has 12 of these triangles, so the area is $5 \cdot 12 = \boxed{60}$.



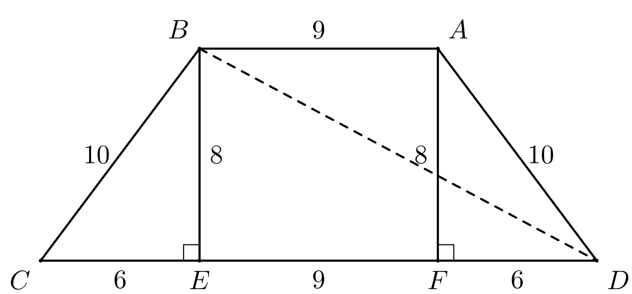
---

2. In trapezoid $ABCD$, $AD = BC$ and $\overline{AB} \parallel \overline{CD}$, with $AB < CD$. Let E be a point on line $\overline{CD}$ such that $\overline{BE} \perp \overline{CD}$. If $BE = 8$, $AB = 9$, and $AD = 10$, compute BD.

Proposed by Yinuo Huang

Answer. 17

Solution.



Let F be the foot of the altitude from A to segment $\overline{CD}$.

Since $AF = BE = 8$ and $AD = 10$, by the Pythagorean Theorem, we have $DF = \sqrt{AD^2 - AF^2} = \sqrt{10^2 - 8^2} = 6$.

Then, $DE = DF + FE = DF + AB = 6 + 9 = 15$. By the Pythagorean Theorem, $BD = \sqrt{BE^2 + DE^2} = \sqrt{8^2 + 15^2} = \boxed{17}$.

---

3. I have a quadratic equation $y = ax^2 + bx + c$ with $a > 0$, and I'm told that one root is three times the other. If the minimum value of y is -2025, what is the y-intercept of this graph?

   Proposed by Lohith Tummala

   Answer. 6075

   Solution. We have $y = a(x - r)(x - 3r)$. The vertex x-coordinate is $2r$, and plugging this in, we get
$$y = a(2r - r)(2r - 3r) = -ar^2 = -2025 \implies ar^2 = 2025$$
   We want to find the y-intercept, aka
$$y = a(0 - r)(0 - 3r) = 3ar^2 = 3 \cdot 2025 = \boxed{6075}$$

---

4. Consider all integers of the form a^2b^4, where a, b are distinct **1-digit** primes. The integer 2025 is on this list, being equal to $5^2 3^4$. What is the smallest integer in this list that is larger than 2025?

   Proposed by Lohith Tummala

   Answer. 2500

   Solution. We could check all the possible products, but let's look more efficiently. Two possible candidates are $7^2 3^4$ and $5^2 7^4$, found by just incrementing each prime up to the next available. Also, we can try incrementing one prime while decrementing the other. Incrementing the squared prime while decrementing the prime raised to the fourth will end up decreasing the number, so let's instead increment the prime raised to the fourth and decrement the squared prime. This transforms $5^2 3^4$ to $3^2 5^4 = 5625$. However, if we decrement the squared prime again to $2^2 5^4$, we get $\boxed{2500}$, which is the lowest after 2025.

---

5. At start of a game of Mario Party, four players each roll a fair 10-sided die to determine turn order, where the game forces all rolls to be distinct. The turn order for the game is from highest roll to lowest roll. If Sharon rolls a 7 and knows nothing about the other rolls, what is the probability they are second in the turn order?

   Proposed by Connor Gordon

   Answer. $\frac{15}{28}$

   Solution. There are $\binom{9}{3} = 84$ possible combinations of the other three dice rolls, and $\binom{3}{1}\binom{6}{2} = 45$ ways for one of the other rolls to be greater than 7 and the other two to be smaller than 7. The desired probability is thus $\boxed{\frac{15}{28}}$.

6. A point is randomly selected in an equilateral triangle of side length 12. The whole triangle is rotated about that point, sweeping out a circle of area A. The lowest possible value of A is $B\pi$, and the highest possible value of A is $C\pi$. Find $B+C$.

   Proposed by Emir Naduvil

   Answer. 192

   Solution. In general, when constructing a point inside any shape, the area of the circle is strictly determined by the maximum distance between that point and any vertex in the shape (in this case, imagine rotating three different segments with varying lengths - they share the same center and are concentric, but the overall area is only determined by the largest radius). To minimize this radius, then the point must necessarily be the circumcenter of the equilateral triangle. Construct a smaller triangle with two vertices of the equilateral triangle and the circumcenter. This forms an isoceles triangle with base angle of 30°. With base 12 (a side length of the triangle), we can split the triangle in half. Then the base is 6; by rules of a 30-60-90 triangle, we have the height as $\frac{6}{\sqrt{3}} = 2\sqrt{3}$. Then, the two congruent sides necessarily have length $\sqrt{6^2 + (2\sqrt{3})^2} = 4\sqrt{3}$, which is the minimum radius. The area of the resultant circle is $\pi r^2 = 48\pi$. For the maximum, simply consider any of the three vertices. Then the distance from one vertex to another is 12, which means the area is maximized at $12^2 \cdot \pi = 144\pi$. The range is then $[48\pi, 144\pi]$, giving us an answer of $48 + 144 = \boxed{192}$

---

7. Shyla and Claire are perfect logicians who always tell the truth. They have the following conversation.

   Shyla: "I'm thinking of a 2-digit number. I came up with it by multiplying two consecutive integers."
   Claire: "What is the sum of the digits of your number?"
   Shyla: "I don't want to say; that will just give it away."
   Claire: "Fine, can you at least tell me if the sum of the digits is prime?"
   Shyla: "That would also give it away."

   What is Shyla's number?

   Proposed by Lohith Tummala

   Answer. 42

   Solution. The possible numbers are 12, 20, 30, 42, 56, 72, 90, and the sums are 3, 2, 3, 6, 11, 9, 9. Alice said the sum gives it away, so we know the sum is unique, so eliminate the sums of 3 and 9, leaving 20, 42, 56. Two of the three remaining sums is prime, so if knowing that the sum is prime gives it away, this shows that the sum is not prime. (If it was prime, we wouldn't know if it was 20 or 56). Thus, the number is $\boxed{42}$.

---

8. Let r, s, t be the roots of $7x^3 + 11x^2 - 13$. Find $\frac{rs}{t} + \frac{rt}{s} + \frac{st}{r}$.

Proposed by Peter Ferolito

Answer. $\frac{22}{7}$

Solution. We COULD bash with Vieta's... but where's the fun in that? Note that $\frac{rs}{t}+\frac{rt}{s}+\frac{st}{r}=\frac{rst}{t^2}+\frac{rst}{s^2}+\frac{rst}{r^2}=\frac{13/7}{t^2}+\frac{13/7}{s^2}+\frac{13/7}{r^2}$ by Vieta's. Additionally, note that rearranging $7x^3+11x^2-13=0$ to $7x^3+11x^2=13$ and dividing by x^2 yields

$$7x+11=\frac{13}{x^2}.$$

Therefore, $x+\frac{11}{7}=\frac{13/7}{x^2}$. We use this to rewrite our sum of roots as $r+s+t+3\cdot\frac{11}{7}=-\frac{11}{7}+3\cdot\frac{11}{7}=\boxed{\frac{22}{7}}$.

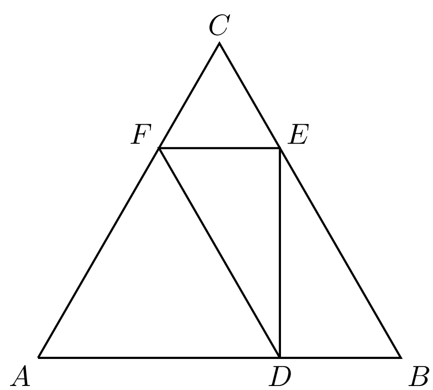
---

9. We have equilateral triangle ABC. There are points D, E, F on sides $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ respectively, such that $\overline{DE}$ is perpendicular to $\overline{AB}$, $\overline{EF}$ is parallel to $\overline{AB}$, and $\overline{DF}$ is parallel to $\overline{BC}$. If the area of $\triangle ABC$ is 2025, what is the area of $\triangle ADF$?

Proposed by Lohith Tummala

Answer. 900

Solution.



Half the battle here is drawing the diagram accurately. Note that since $\overline{DF}$ is parallel to $\overline{BC}$, $\triangle ADF$ is equilateral. By similar logic, $\triangle CFE$ is equilateral. Also, note that $\triangle BDE$ is a $30-60-90$ triangle.

We can deduce that $\angle DFE = 60°$ and $\angle EDF = 30°$. From this, we deduce that $\triangle DEF$ is congruent to $\triangle BDE$ since we can prove that all the angles correspond and the two triangles share a side. Thus, $AD = BE = 2\cdot DB$. Thus, AD is two-thirds the length of DB, and since $\triangle ADF$ and $\triangle ABC$ are similar, the area ratios are obtained by squaring the side ratios, so the area of $\triangle ADF$ is 4/9 the area of $\triangle ABC$.

$$4/9\cdot 2025=\boxed{900}$$

10. Given that $\lfloor x \rfloor$ equals the greatest integer less than or equal to x, find the smallest (possibly negative) integer y that satisfies:

$$\lfloor \log_2(\lfloor \log_3(y^2 + 2y + 1) \rfloor) \rfloor = 1$$

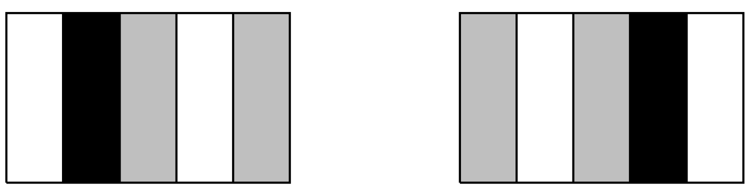
Proposed by Ankita Varigonda

Answer. -9

Solution. The floor of $\log_2$ of a value is equal to 1 when the logarithm evaluates to a number between 1 (inclusive) and 2 (exclusive). This occurs only when the value is between $2^1 = 2$ and $2^2 = 4$. Thus, we have $2 \leq \lfloor \log_3(y^2 + 2y + 1) \rfloor < 4$. Using the same logic, $y^2 + 2y + 1$ must then be between 3^2 (inclusive) and 3^4 (exclusive): $9 \leq y^2 + 2y + 1 < 81$. Since $y^2 + 2y + 1$ factors to $(y + 1)^2$, we have $9 \leq (y + 1)^2 < 81$. The smallest value $y + 1$ can then take is -9 as with -10, the square would equal 81. Therefore, $y = \boxed{-9}$.

---

11. We paint 5 vertical stripes on a glass door, using a combination of the colors red, blue, and yellow. The glass door is see-through from both sides for any color. How many **distinct** color patterns are there such that no two patterns can be obtained by simply walking to the other side of the glass door? (For example, the two patterns below are NOT distinct, as we can walk to the other side of one to see the other.)



Proposed by Baili Jiang

Answer. 135

Solution. There are a total of $3^5 = 243$ total colorings. Of those $3^3 = 27$ are symmetric, since the first and fifth, second and fourth, and third stripes have 3 color options each to be considered symmetric. Therefore, there are $243 - 27 = 216$ total asymmetric colorings. Taking any asymmetric coloring gets you a new design after walking to the other side of the glass door, so really, we only have $\dfrac{216}{2} = 108$ distinct assymetric colorings. Add the symmetric colorings back, and we get $108 + 27 = \boxed{135}$ distinct total colorings.

---

12. Let m and n be two real numbers such that

$$\frac{2}{n} + m = 9, \quad \frac{2}{m} + n = 1$$

Find the sum of all possible values of m and all possible values of n.

Proposed by BangTam Ngo

Answer. 10

Solution. Multiply both equations so you get

$$\frac{4}{nm} + nm + 4 = 9$$

Then solve for nm like a quadratic equation so that you get

$$(nm - 4)(nm - 1) = 0$$

If $nm = 4$, then we can plug in $m = 4/n$ into the original equations to get

$$n = \frac{2}{3}, m = 6$$

If $nm = 1$, then similarly, we can get

$$n = \frac{1}{3}, m = 3$$

Add them all up together

$$\frac{1}{3} + \frac{2}{3} + 6 + 3 = \boxed{10}$$

---

13. A bug is on the corner of a cube of length 2 inches and can take *steps* on the cube. A step must be of length 1 inch and the step must be parallel to some edge of the cube (aka the ant can't travel diagonally on the cube). How many 6-step paths can it take to the far opposite corner of the cube?

Proposed by John Li

Answer. 54

Solution. Notice that we can represent each path as a sequence of unit steps along the lattice from $(0, 0, 0)$ to $(2, 2, 2)$, but with the extra condition that the path cannot go inside the cube. Every path that goes inside the cube passes through the point $(1, 1, 1)$, so we are counting the paths from $(0, 0, 0)$ to $(2, 2, 2)$ that do not go through $(1, 1, 1)$. Each such path can be represented as a string of two "up", two "right", and two "in" moves. Thus, the total number of paths not excluding the illegal ones is $\frac{6!}{2!2!2!} = 90$. The ones that pass through $(1, 1, 1)$ can be thought of as the combination of two separate paths, from $(0, 0, 0)$ to $(1, 1, 1)$ and from $(1, 1, 1)$ to $(2, 2, 2)$, which there are $\frac{3!}{1!1!1!} \cdot \frac{3!}{1!1!1!} = 36$ of. Thus, the final answer is $90 - 36 = \boxed{54}$.

---

14. Find the remainder when the 2997-digit number $999, 998, 997, 996, \ldots, 002, 001$ is divided by 1001.

Proposed by Henry Zheng

Answer. 500

Solution. Notice that $999,998,998,\ldots,002,001 = \sum_{n=0}^{998}(n+1)\times 10^{3n}$. Therefore, when we take the mods, we get $1-2+3-4+5-6+\ldots-998+999 = \boxed{500} \pmod{1001}$.

---

15. Spheres S_1, S_2, and S_3 have radii 1, 2, and 3 respectively, and are all internally tangent at a single point. A plane P intersects each of the spheres at a circle. If the intersection of P and S_1 has area π and the intersection of P and S_3 has area 7π, compute the area of the intersection of P and S_2.

Proposed by Connor Gordon

Answer. $\frac{7\pi}{2}$

Solution. Note that the center of S_2 is the midpoint of the centers of S_1 and S_3. As such, the distance from S_2 to P is the average of the distances from S_1 and S_3 to P.

We can compute the distances from S_1 and S_3 to P to be $\sqrt{1^2-1} = 0$ and $\sqrt{3^2-7} = \sqrt{2}$ respectively. It thus follows that the distance from P to S_2 is $\frac{\sqrt{2}}{2}$, and thus the desired area is $\pi(2^2 - (\frac{\sqrt{2}}{2})^2) = \boxed{\frac{7\pi}{2}}$.

---

16. On a grid, Mario starts at $(0,0)$, moving up or right 1 unit per turn with equal probability. Bowser starts at $(4,4)$, moving left or down 1 unit, making 3 moves per turn, each equally likely. On a turn, Mario and Bowser execute their moves at the same time. Mario is caught if they occupy the same point after their turn; he is safe if Bowser can never reach his position (either left or below Mario). Neither character is confined to this 4×4 grid. What is the probability that Mario is **safe**? (For example, after a single turn, Mario can possibly end up at $(1,0)$, and Bowser can possibly end up at $(3,2)$.)

Proposed by Eugene Hwang

Answer. $\frac{93}{128}$

Solution. Let Mario's starting point be (0,0) and Bowser's initial position to be (4,4).

Let T be the turn at which Bowser and Mario meet. Let R be the amount of right moves that Mario makes. Then, at turn T, Mario would be at position (R, T - R).

Let L be the amount of left moves that Bowser makes. Then, at turn T, Bowser would be at position (4 - L, 4 - (3T - L)). Since Mario and Bowser are at the same point, $R = 4 - L$. We can then see:

$$\begin{aligned}
T - R &= 4 - (3T - L) \\
T - (4 - L) &= 4 - 3T + L \\
T - 4 + L &= 4 - 3T + L \\
4T &= 8 \\
T &= 2
\end{aligned}$$

Mario and Bowser will meet at the 2nd turn, at which the only possible points are X(0, 2), Y(1, 1), and Z(2, 0).

$P(\text{Mario is not safe}) = P(\text{M} \to \text{X})P(\text{B} \to \text{X}) + P(\text{M} \to \text{Y})P(\text{B} \to \text{Y}) + P(\text{M} \to \text{Z})P(\text{B} \to \text{Z})$
$= \frac{\binom{2}{0}}{2^2} * \frac{\binom{6}{4}}{2^6} + \frac{\binom{2}{1}}{2^2} * \frac{\binom{6}{3}}{2^6} + \frac{\binom{2}{2}}{2^2} * \frac{\binom{6}{2}}{2^6} = \frac{35}{128}$

$P(\text{Mario is safe}) = 1 - P(\text{Mario is not safe}) = \boxed{\frac{93}{128}}$.

---

17. For any two nonnegative integers $A, B < 2^{10}$, define a function $\&(A, B)$ so that given the unique binary representations $A = a_0 + a_1 \cdot 2 + a_2 \cdot 2^2 + \cdots + a_9 \cdot 2^9$ and $B = b_0 + b_1 \cdot 2 + b_2 \cdot 2^2 + \cdots + b_9 \cdot 2^9$, $\&(A, B) = \sum_{i=0}^{9}(a_i \cdot b_i)2^i$. For example, if $A = 5$ (0000000101 in binary) and $B = 12$ (0000001100 in binary), $\&(A, B) = 4$ (0000000100 in binary). Find the sum of all integers X with $0 < X < 2^{10}$ such that $\&(X, X - 1) = 0$.

Proposed by Alan Abraham

Answer. 1023

Solution. We find the first 1 in the binary representation of X (starting from the right). Let's say that is at position i. Then X can be represented as $1 \times 2^i + \sum_{j=i+1}^{9} x_j 2^j$. After performing the subtraction $X - 1$, we see $X - 1$ can be represented as $1 + 2 + \cdots + 2^{i-1} + \sum_{j=i+1}^{9} x_j 2^j$. So this means $\&(X, X - 1) = \sum_{j=i+1}^{9} x_j 2^j$. Since we want this to be 0, this implies that X must only have one 1 in its binary representation.

So the only possible X that satisfy the condition are the powers of 2. The sum of all powers of 2 less than 2^{10} is $\boxed{1023}$.

---

18. Call a 4-dight number $\overline{abcd}$ *interesting* if $(\overline{ab} + \overline{cd})^2 = \overline{abcd}$. How many 4-digit interesting numbers are there? (Here, $\overline{abcd}$ means the 4-digit number using the digits a, b, c, d. Also, a cannot be 0.)

Proposed by Gorden Jin

Answer. 3

Solution. Let's represent the 4-digit number as $100x + y$, where x is a two-digit number and y can be either two-digit or one-digit. We have that

$$100x + y = (x + y)^2 = (x + y)(x + y)$$

$$\implies 99x + x + y = (x + y)(x + y) \implies 99x = (x + y)(x + y - 1)$$

At this point, notice that $99x$ must be the product of two consecutive integers. Since the upper bound of x is 99, each of these consecutive numbers must also be bounded by 99. One of these numbers must be a multiple of 11, and one of these numbers must be a multiple of 9 (possibly the same number). If we analyze the two-digit multiples of 11 and check which ones is or is next to a multiple of 9, we have the following:

$$10, 11, 12 \implies no$$

$$21, 22, 23 \implies no$$

$$32, 33, 34 \implies no$$

$$43, 44, 45 \implies yes, (44, 45)$$

$$54, 55, 56 \implies yes, (54, 55)$$

$$65, 66, 67 \implies no$$

$$76, 77, 78 \implies no$$

$$87, 88, 89 \implies no$$

$$98, 99, 100 \implies yes, (99)$$

We have three candidates here. Let's look at each one:

$$99x = 45 \cdot 44 \implies x = 20 \implies y = 25$$

$$99x = 55 \cdot 54 \implies x = 30 \implies y = 25$$

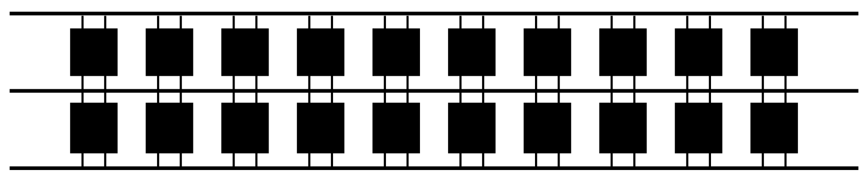
$$99x = 99 \cdot 98 \implies x = 98 \implies y = 1$$

We have $2025, 3025, 9801$, giving $\boxed{3}$ interesting numbers.

19. A bridge consists of 10 steps. At each step there are two platforms, one of which is safe to walk on and the other is is unsafe (causes you to instantly lose). The identity of each safe side is chosen uniformly at random, and is initially unknown.

Players take turns attempting to cross the bridge. At each step, players must choose one of the two platforms to walk on. If they ever step on an unsafe side, they will be removed from the game. Assume each player in line is perfectly rational and remembers each players steps. A player wins if they are the first one to make it across the bridge only using correct steps.

If there are 11 people in line, what is the probability that the 4th person is the first to cross successfully?



Proposed by Michael Duncan

Answer. $\frac{15}{128}$

Solution. The key is that we can view this game instead as 10 fair coin flips. Note that each player has a 1/2 chance of stepping on the correct side at each step, but after they fall off the next player gets to continue where this player left off. Also each possible game state is equivalent to a flip of 10 fair coins so we have a bijection. So the probability you win is the probability that there are 3 tails, or $\binom{10}{3} \cdot \frac{1}{2^{10}} = \boxed{\frac{15}{128}}$

---

20. Let C be the graph of

$$\frac{(x-20)^2}{25} - \frac{(y-26)^2}{16} = 1$$

on the xy-plane. Define C' as the graph when C is rotated $45°$ counter-clockwise about the origin. C' intersects the y-axis at two distinct points $(0, y_1)$ and $(0, y_2)$. Compute $y_1 + y_2$.

Proposed by Yinuo Huang

Answer. $\frac{220\sqrt{2}}{3}$

Solution. Instead of rotating C $45°$ counter-clockwise, consider rotating the y-axis $45°$ clockwise instead. This creates the same scenario in the problem, except the whole plane is rotated by $45°$, where the line $y = x$ in our method corresponds to the y-axis in the original problem, and C in our methods corresponds to C' in the original problem.

We can then easily compute the intersections between C and $y = x$. Plugging in $y = x$ into the original equation, we have

$$\begin{aligned}\frac{(x-20)^2}{25}-\frac{(x-26)^2}{16}&=1\\16(x-20)^2-25(x-26)^2&=400\\-9x^2+660x+k&=0\end{aligned}$$

where k is some constant not necessary to solve the problem, so we don't need to expand out the constant terms.

Suppose the two solutions are x_1' and x_2'. To answer the original problem, we need to consider the distance between the origin and the points (x_1', x_1') and (x_2', x_2'), which are $y_1 = \sqrt{2}x_1'$ and $y_2 = \sqrt{2}x_2'$. The desired sum $y_1 + y_2$ is then equal to $\sqrt{2}(x_1' + x_2')$.

Since the problem implies the roots are distinct, by Vieta's formula, $x_1' + x_2' = -\dfrac{660}{-9} = \dfrac{220}{3}$. Therefore,

$$y_1 + y_2 = \sqrt{2}(x_1' + x_2') = \sqrt{2}\cdot\frac{220}{3} = \boxed{\frac{220\sqrt{2}}{3}}.$$

In fact, $k = -10900$, and C' intersects the y-axis at $\left(0, \frac{110\sqrt{2}\pm 20\sqrt{6}}{3}\right)$.

---

21. (Estimation) Define the function $f(n)$ as follows: let a be the greatest integer such that 2^a divides n; then, $f(n) = \dfrac{n}{2^a}$. For example, $f(12) = \dfrac{12}{2^2} = 3$. Estimate

$$\sum_{x=1}^{100}\sum_{y=x}^{100} f(x+y).$$

Proposed by Yinuo Huang

Answer. 338350

Solution. Let us consider the integers in the set $\{1, 2, \ldots, 100\}$.

For a positive integer n, let a be the greatest integer such that 2^a divides n as defined in the function f.

Note that $\frac{1}{2}$ of these integers are odd, so $a = 0$ for about $\frac{1}{2}$ of the numbers; about $\frac{1}{4}$ of these integers are multiples of 2 but not 4, so $a = 1$ for about $\frac{1}{4}$ of the numbers, etc.

In this way, we can see that for all $m = 0, 1, 2, \ldots$, we have $a = m$ for about 2^{-m-1} of the integers.

Therefore, by the formula for the sum of a geometric series,

$$f(n) \approx n\cdot\sum_{m=0}^{\infty}\frac{2^{-m-1}}{2^m} = n\cdot\sum_{m=0}^{\infty}2^{-2m-1} = n\cdot\frac{1/2}{1-1/4} = \frac{2}{3}\cdot n.$$

Therefore, a good approximation for the function f is $f(n) \approx \dfrac{2}{3}\cdot n$.

Using this approximation, we have

$$\sum_{x=1}^{100}\sum_{y=x}^{100} f(x+y) \approx \sum_{x=1}^{100}\sum_{y=x}^{100} \frac{2}{3}(x+y) = \frac{2}{3}\cdot\sum_{x=1}^{100}\sum_{y=x}^{100}(x+y).$$

Note that in the double summation, for each $k \in \{1, 2, \ldots, 100\}$, k is added exactly 101 times. Thus this expression is equal to

$$\frac{2}{3}\cdot 101 \cdot \sum_{k=1}^{100} k = \frac{(101)(100)(101)}{3} = \frac{1020100}{3} \approx 340033.$$

This is enough to earn $\frac{338350}{340033} \approx 0.995$ points.

The following Python program computes the exact answer $\boxed{338350}$:

```python
def f(n):
    while n % 2 == 0:
        n //= 2
    return n

print(sum(f(x + y) for x in range(1, 101) for y in range(x, 101))) # 338350
```