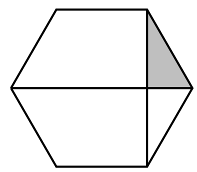
2025 Individual Round

Instructions

1. Do not look at the test before the proctor starts the round.
2. This test consists of 20 short-answer problems to be solved in 60 minutes. The final estimation question will be used to break ties.
3. No computational aids other than pencil/pen are permitted.
4. Write your name, team name, and contestant ID on your answer sheet.
5. Write your answers in the corresponding lines on your answer sheet.
6. Answers must be reasonably simplified.
7. If you believe that the test contains an error, submit your protest on the CMWMC Discord channel.

Individual Round

1. The area of the shaded region is 5. What is the area of the entire hexagon, given that it is regular?



2. In trapezoid $ABCD$, $AD = BC$ and $\overline{AB} \parallel \overline{CD}$, with $AB < CD$. Let E be a point on line $\overline{CD}$ such that $\overline{BE} \perp \overline{CD}$. If $BE = 8$, $AB = 9$, and $AD = 10$, compute BD.

3. I have a quadratic equation $y = ax^2 + bx + c$ with $a > 0$, and I'm told that one root is three times the other. If the minimum value of y is -2025, what is the y-intercept of this graph?

4. Consider all integers of the form a^2b^4, where a, b are distinct **1-digit** primes. The integer 2025 is on this list, being equal to $5^2 3^4$. What is the smallest integer in this list that is larger than 2025?

5. At start of a game of Mario Party, four players each roll a fair 10-sided die to determine turn order, where the game forces all rolls to be distinct. The turn order for the game is from highest roll to lowest roll. If Sharon rolls a 7 and knows nothing about the other rolls, what is the probability they are second in the turn order?

6. A point is randomly selected in an equilateral triangle of side length 12. The whole triangle is rotated about that point, sweeping out a circle of area A. The lowest possible value of A is $B\pi$, and the highest possible value of A is $C\pi$. Find $B + C$.

7. Shyla and Claire are perfect logicians who always tell the truth. They have the following conversation.

   Shyla: "I'm thinking of a 2-digit number. I came up with it by multiplying two consecutive integers."
   Claire: "What is the sum of the digits of your number?"
   Shyla: "I don't want to say; that will just give it away."
   Claire: "Fine, can you at least tell me if the sum of the digits is prime?"
   Shyla: "That would also give it away."

   What is Shyla's number?

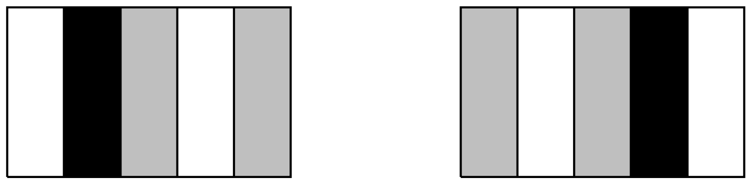
8. Let r, s, t be the roots of $7x^3 + 11x^2 - 13$. Find $\frac{rs}{t} + \frac{rt}{s} + \frac{st}{r}$.

9. We have equilateral triangle ABC. There are points D, E, F on sides $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ respectively, such that $\overline{DE}$ is perpendicular to $\overline{AB}$, $\overline{EF}$ is parallel to $\overline{AB}$, and $\overline{DF}$ is parallel to $\overline{BC}$. If the area of $\triangle ABC$ is 2025, what is the area of $\triangle ADF$?

10. Given that $\lfloor x \rfloor$ equals the greatest integer less than or equal to x, find the smallest (possibly negative) integer y that satisfies:

$$\lfloor \log_2(\lfloor \log_3(y^2 + 2y + 1) \rfloor) \rfloor = 1$$

11. We paint 5 vertical stripes on a glass door, using a combination of the colors red, blue, and yellow. The glass door is see-through from both sides for any color. How many **distinct** color patterns are there such that no two patterns can be obtained by simply walking to the other side of the glass door? (For example, the two patterns below are NOT distinct, as we can walk to the other side of one to see the other.)



12. Let m and n be two real numbers such that

$$\frac{2}{n} + m = 9, \quad \frac{2}{m} + n = 1$$

Find the sum of all possible values of m and all possible values of n.

13. A bug is on the corner of a cube of length 2 inches and can take *steps* on the cube. A step must be of length 1 inch and the step must be parallel to some edge of the cube (aka the ant can't travel diagonally on the cube). How many 6-step paths can it take to the far opposite corner of the cube?

14. Find the remainder when the 2997-digit number $999, 998, 997, 996, \ldots, 002, 001$ is divided by 1001.

15. Spheres S_1, S_2, and S_3 have radii 1, 2, and 3 respectively, and are all internally tangent at a single point. A plane P intersects each of the spheres at a circle. If the intersection of P and S_1 has area π and the intersection of P and S_3 has area 7π, compute the area of the intersection of P and S_2.

16. On a grid, Mario starts at $(0, 0)$, moving up or right 1 unit per turn with equal probability. Bowser starts at $(4, 4)$, moving left or down 1 unit, making 3 moves per turn, each equally likely. On a turn, Mario and Bowser execute their moves at the same time. Mario is caught if they occupy the same point after their turn; he is safe if Bowser can never reach his position (either left or below Mario). Neither character is confined to this 4×4 grid. What is the probability that Mario is **safe**? (For example, after a single turn, Mario can possibly end up at $(1, 0)$, and Bowser can possibly end up at $(3, 2)$.)

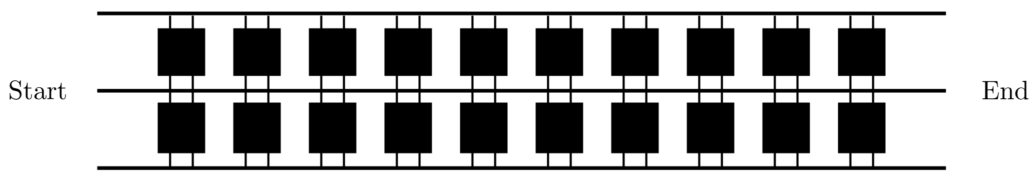
17. For any two nonnegative integers $A, B < 2^{10}$, define a function $\&(A, B)$ so that given the unique binary representations $A = a_0 + a_1 \cdot 2 + a_2 \cdot 2^2 + \cdots + a_9 \cdot 2^9$ and $B = b_0 + b_1 \cdot 2 + b_2 \cdot 2^2 + \cdots + b_9 \cdot 2^9$, $\&(A, B) = \sum_{i=0}^{9} (a_i \cdot b_i) 2^i$. For example, if $A = 5$ (0000000101 in binary) and $B = 12$ (0000001100 in binary), $\&(A, B) = 4$ (0000000100 in binary). Find the sum of all integers X with $0 < X < 2^{10}$ such that $\&(X, X - 1) = 0$.

18. Call a 4-dight number $\overline{abcd}$ *interesting* if $(\overline{ab} + \overline{cd})^2 = \overline{abcd}$. How many 4-digit interesting numbers are there? (Here, $\overline{abcd}$ means the 4-digit number using the digits a, b, c, d. Also, a cannot be 0.)

19. A bridge consists of 10 steps. At each step there are two platforms, one of which is safe to walk on and the other is is unsafe (causes you to instantly lose). The identity of each safe side is chosen uniformly at random, and is initially unknown.

Players take turns attempting to cross the bridge. At each step, players must choose one of the two platforms to walk on. If they ever step on an unsafe side, they will be removed from the game. Assume each player in line is perfectly rational and remembers each players steps. A player wins if they are the first one to make it across the bridge only using correct steps.

If there are 11 people in line, what is the probability that the 4th person is the first to cross successfully?



20. Let C be the graph of

$$\frac{(x-20)^2}{25} - \frac{(y-26)^2}{16} = 1$$

on the xy-plane. Define C' as the graph when C is rotated 45° counter-clockwise about the origin. C' intersects the y-axis at two distinct points $(0, y_1)$ and $(0, y_2)$. Compute $y_1 + y_2$.

21. (Estimation) Define the function $f(n)$ as follows: let a be the greatest integer such that 2^a divides n; then, $f(n) = \frac{n}{2^a}$. For example, $f(12) = \frac{12}{2^2} = 3$. Estimate

$$\sum_{x=1}^{100} \sum_{y=x}^{100} f(x+y).$$