2025 CMWMC Guts Round Solutions

1. How many positive factors of 2025 are either perfect squares or perfect cubes?

   Proposed by Lohith Tummala

   Answer. 7

   Solution. The prime factorization of 2025 is $5^2 \cdot 3^4$. We find the perfect squares by taking every factor with even exponents. This gives $5^0 \cdot 3^0$, $5^2 \cdot 3^0$, $5^0 \cdot 3^2$, $5^2 \cdot 3^2$, $5^0 \cdot 3^4$, $5^2 \cdot 3^4$. Doing a similar thing with perfect cubes, we get $5^0 \cdot 3^0$ and $5^0 \cdot 3^3$. Given that $5^0 \cdot 3^0 = 1$ is in both lists and we shouldn't double-count it, we have a total of $\boxed{7}$ perfect squares or perfect cubes.

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2. How many ordered pairs of non-negative integers (a, b) satisfy $\sqrt{a} + \sqrt{b} = \sqrt{a + b + 104}$?

   Proposed by Minerva You

   Answer. 15

   Solution.

$$\begin{aligned}
&\sqrt{a} + \sqrt{b} = \sqrt{a + b + 104} \\
\Longrightarrow\ &(\sqrt{a} + \sqrt{b})^2 = (\sqrt{a + b + 104})^2 \\
\Longrightarrow\ &a + 2\sqrt{ab} + b = a + b + 104 \\
\Longrightarrow\ &2\sqrt{ab} = 104 \\
\Longrightarrow\ &ab = 52^2 \\
\Longrightarrow\ &ab = 2^4 13^2
\end{aligned}$$

   Finally, notice that $2^4 13^2$ has $(4+1)(2+1) = \boxed{15}$ non-negative factors, which gives us the answer.

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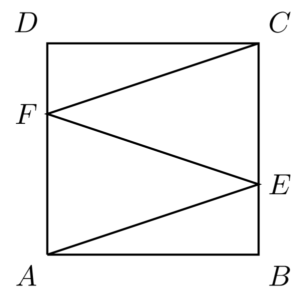
3. In unit square $ABCD$, let E be on BC and F be on AD such that triangles AEF and ECF are congruent and isosceles, with bases of AF and CE. What is the length of the path of $AE + EF + FC$?

   Proposed by Alan Abraham

   Answer. $\sqrt{10}$

   Solution.



Triangles AEF and ECF being isosceles and congruent implies that $AE = EF = FC$. The isosceles condition further implies that $AF = 2 \cdot EB$, so if $EB = x$, $AF = EC = 2x$, so $BC = 3x = 1$, implying $x = \frac{1}{3}$.

The length of the path is just $3 \cdot AE = 3\sqrt{1^2 + \left(\frac{1}{3}\right)^2} = 3\sqrt{\frac{10}{9}} = \boxed{\sqrt{10}}$.

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4. For a three-letter string, define the "score" to be the position of the word in an alphabetically-sorted list of all three-letter strings. For example, "aaa" would have a score of 1, "aad" would have a score of 4, and "zzz" would have a score of 26^3. My friend wrote the word "dig" on a napkin and asked me to compute the score, but a jester sneakily erased a random letter and wrote a **different** random letter in its place (eg. "dig" to "dxg"). What is the probability that the score went up?

   Proposed by Lohith Tummala

   Answer. $\frac{58}{75}$

   Solution. The jester had a $\frac{1}{3}$ chance of choosing each letter position. If the jester chose the "d", there's 25 letters to randomize to, and three of them lead to a lower score ("a", "b", and "c"), meaning 22 letters lead to a higher score. By similar logic, if the jester chose the "i", there are 17 letters that lead to a higher score, and with the "g", there are 19 letters that lead to a higher score. We can then compute the probability:

$$\frac{1}{3}\left(\frac{22}{25}\right) + \frac{1}{3}\left(\frac{17}{25}\right) + \frac{1}{3}\left(\frac{19}{25}\right) = \boxed{\frac{58}{75}}$$

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5. The three sets A, B, and C each have 400 elements. Each intersection of two sets has 150 elements. The intersection of all three sets has 80 elements. How many elements are in at least one set?

   Proposed by Baili Jiang

Answer. 830

Solution. We can initially claim it is just $3 \cdot 400$ since each set has 400 elements, but we definitely double count the elements in each set intersection. For example, an element in both set A and B was counted twice when it should be counted once. We can then subtract: $3 \cdot 400 - 3 \cdot 150 = 750$. But now, imagine an element that is in all three sets. It was counted three times when we did $3 \cdot 400$, but when we subtracted the three intersections, the element was in all three intersections, so it is never counted now, so we must add back the 80 elements in all three sets to get $\boxed{830}$.

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6. Let OA_1 be a line segment with side length 1. From the point A_i, we generate the next point A_{i+1} in the following manner: we construct A_{i+1} such that $A_{i+1}A_i \perp A_iO$ and $\overline{A_{i+1}A_i} = i+1$. Find the length of OA_{10}.

   Proposed by Henry Zheng

   Answer. $\sqrt{385}$

   Solution. Note that $OA_1 = 1$ and $A_2A_1 = 2$, making $OA_2 = \sqrt{1^2+2^2}$. Also note that since $A_3A_2 = 3$ and $A_3A_2 \perp OA_2$, $OA_3 = \sqrt{1^2+2^2+3^2}$. This pattern continues, up until

$$OA_{10} = \sqrt{1^2+2^2+\cdots+10^2} = \sqrt{\frac{10 \cdot 11 \cdot 21}{6}} = \boxed{\sqrt{385}}$$

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7. Define the function $f(n)$ to take the largest integer k such that $n^k \mid n!$. Find

$$\sum_{i=2}^{6} f(i!).$$

   Proposed by Henry Zheng

   Answer. 216

   Solution. We need to find $f(2)+f(6)+f(24)+f(120)+f(720)$. Let's find each.
   For $f(2)$, the largest power of 2 that divides 2! is 2, so $f(2) = 1$.
   For $f(6)$, the largest power of 6 that divides $6! = 720$ is $6^2 = 36$, so $f(2) = 2$.
   For $f(24)$, to find the largest power of 24 that divides 24!, we have to analyze the prime factorization of 24!, specifically the powers of 2 and 3. Legendre's formula helps with this. We can use it to get $12+6+3+1 = 22$ powers of 2 and $8+2 = 10$ powers of 3. Since $24 = 2^3 3^1$, we can fit 7 powers of 24, which will get us to $2^{21}3^7$, which is still a factor of $24! = 2^{22}3^{10} \cdot k$. Thus, we have $f(24) = 7$. We follow a similar method for the rest of the cases.
   For $f(120)$, we have $120! = 2^{116}3^{58}5^{28} \cdot k$ and $120 = 2^3 3^1 5^1$, so we can fit 28 powers of 120, which gets us to $2^{72}3^{24}5^{24}$. Thus, $f(120) = 28$.
   For $f(720)$, we have $720! = 2^{716}3^{356}5^{178} \cdot k$ and $720 = 2^4 3^2 5^1$, so we can fit 178 powers of 720, which gets us to $2^{712}3^{356}5^{178}$. Thus, $f(720) = 178$.
   Thus,

$$f(2)+f(6)+f(24)+f(120)+f(720) = 1+2+7+28+178 = \boxed{216}$$

8. I have a digital gate (number machine) Z that accepts three inputs and gives one output. Each input is allowed to be either 0 or 1. The output is a 1 if and only if exactly two inputs are 1. Otherwise, the output is 0.

   I have seven inputs a, b, c, d, e, f, g. Inputs a, b, c feed into a Z gate and e, f, g feed into another Z gate. The two outputs and d are fed as inputs to a third Z gate, which gives a final output.

   How many input strings for $abcdefg$ are there such that the final output is a 1?

   Proposed by Lohith Tummala

   Answer. 39

   Solution. Let's consider two cases:

   Case $d = 0$: this means that the two lower Z gates must produce a 1. For a, b, c, there are three ways to produce a 1 for their Z gate, same for e, f, g, giving 9 ways.

   Case $d = 1$: this means that one Z gate gives a 1, and one gives a 0. To give a 0, we just need the three inputs to not be one of the three ways to produce a 1: $8 - 3 = 5$. $5 \cdot 3 \cdot 2$ (since there are two choices for Z gates to give a 1 vs 0) $= 30$.

   Answer: $30 + 9 = \boxed{39}$

9. Alice and Bob received a unit circle cookie from their teacher for being such excellent students. However, they aren't so excellent at splitting cookies. To share the cookie, they each pick a point on the circle's edge uniformly at random and cut a straight line from one point to the other. What is the expected difference in perimeters between the big and small cookie chunks?

   Proposed by Nathan Ye

   Answer. π

   Solution. Let x° be the degree measure of the smaller chunk. This means that larger chunk has degree measure $360^\circ - x^\circ$. The perimeter of the two chunks is given by

$$2\pi \cdot \frac{x}{360} + d, \quad 2\pi \cdot \frac{360 - x}{360} + d$$

   , where d is the length of the straight-line chord of each chunk. The positive difference between the perimeters is

$$(2\pi \cdot \frac{360 - x}{360} + d) - (2\pi \cdot \frac{x}{360} + d) = 2\pi \cdot \frac{360 - 2x}{360}$$

Since x is between 0 and 180 and x is uniformly random in this interval, the average value of x is $90°$, so if we plug this in, we get

$$2\pi \cdot \frac{360 - 2(90)}{360} = \boxed{\pi}$$

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10. Find the sum

$$\sum_{n=1}^{\infty} \frac{3n}{(n+1)!}$$

Proposed by Wilson Pan

Answer. 3

Solution. Factoring out the 3, let

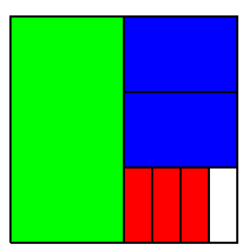
$$S = \sum_{n=1}^{\infty} \frac{n}{(n+1)!}$$

then consider coloring a 1×1 square. The terms are $\frac{1}{1 \cdot 2}, \frac{2}{1 \cdot 2 \cdot 3}, \frac{3}{1 \cdot 2 \cdot 3 \cdot 4} \cdots$

Let's rewrite each term as

$$\frac{1}{1} \cdot \frac{1}{2}, \frac{1}{1 \cdot 2} \cdot \frac{2}{3}, \frac{1}{1 \cdot 2 \cdot 3} \cdot \frac{3}{4} \cdots$$

For this square, the first term demonstrates that of this full, unpainted square, we split it into two and paint one half (green). Then, the second term describes finding the unpainted half, splitting this half into thirds, then painting two of the thirds (blue). The third term is locating the unpainted sixth of the grid, splitting it into fourths, and painting three of the four quarters (red). This process continues until we color the entire square after infinite steps. So $S = 1$ and $3S = \boxed{3}$.



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11. I start with the number 32. I have three possible operations to transform this number into another number: subtract 3, add 4, and divide by 5 (only possible if the number is divisible by 5). What is the least number of operations needed to get to 0?

Proposed by Lohith Tummala

Answer. 6

Solution. The optimal solution is to subtract 3, subtract 3, add 4, divide by 5, subtract 3, then subtract 3, to get $\boxed{6}$. For some intuition, since we have to end at 0, the second to last value is 3, so we need to get from 32 to 3. We cover a lot of distance by using a divide, so we optimize for using those. The division of 30 to 6 happens to be the useful one here.

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12. Jerry starts at 0 on the real number line. He tosses a fair coin exactly 8 times. When he gets heads, he moves 1 unit in the positive direction; when he gets tails, he moves 1 unit in the negative direction. The probability that he reaches 4 at some time during this process is $\frac{m}{n}$, where m and n are relatively prime positive integers. (For example, he succeeds if his sequence of tosses is HTHHHHHH since he reaches 4 after the 6th flip.)

Proposed by Baili Jiang

Answer. 151

Solution. First, we need to figure out the probability $\dfrac{m}{n}$. To do that, we do casework based on the total number of heads flipped.

Case 1 - There are 8, 7, or 6 heads: If this is the case, Jerry is guaranteed to reach $(+4)$ at one point. There are a total of $\binom{8}{0}+\binom{8}{1}+\binom{8}{2}=37$ sequences of 8 flips that satisfy this case, each with probability $(\dfrac{1}{2})^8=\dfrac{1}{256}$. So this case has total probability $\dfrac{37}{256}$

Case 2 - There are exactly 5 heads: 5 heads means there are 3 tails as well. Since $5-3=2<4$, the 5 heads must come early before the tails have a chance to have much of an effect. We can split this into two subcases.

If the 5 heads all appear in the first 6 flips, then we are guaranteed to reach 4. This has probability $6\times\dfrac{1}{256}$. If only 4 heads appear in the first 6 flips, the only way to reach 4 is if all the heads are at the beginning, which has probability $2\times\dfrac{1}{256}$. So the total probability is $\dfrac{8}{256}$.

Case 3 - There are exactly 4 heads: The only way for this to work is if all 4 heads appear at the start of the sequence, which has probability $\dfrac{1}{256}$.

Note that if there are less than 4 heads flipped, it is impossible to reach 4, so these 3 cases are exhaustive, so the total probability is

$$\frac{37+8+1}{256}=\frac{46}{256}=\frac{23}{128}$$

Therefore $m=23$ and $n=128$, and $m+n=\boxed{151}$.

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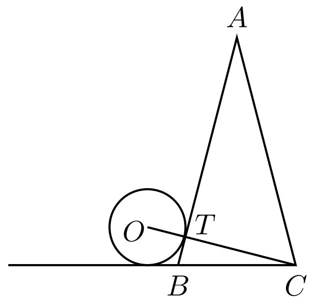
13. Isosceles triangle ABC has base BC length $2\sqrt{3}$ and side length $4\sqrt{3}$. Circle ω with center O is tangent to AB and line BC. Let r be the radius of ω. Let T be the point of tangency of ω and AB. If O, T, and C are collinear and $r^2=\frac{a}{b}$, where a and b are relatively prime positive integers, find $a+b$.



Proposed by BangTam Ngo

Answer. 9

Solution. Let M be the midpoint of BC. Notice that $\triangle BTC \sim \triangle BMA$:

$$\frac{BC}{AB} = \frac{2\sqrt{3}}{4\sqrt{3}} = \frac{BT}{BM} = \frac{BT}{\sqrt{3}}$$

$$\frac{1}{2} = \frac{BT}{\sqrt{3}} \implies BT = \frac{\sqrt{3}}{2}$$

Let Y be the foot of the altitude from O to $\overleftrightarrow{BC}$, and notice that $\triangle BTC \sim \triangle OYC$:

$$\frac{BC}{OC} = \frac{2\sqrt{3}}{\frac{3\sqrt{5}}{2} + r} = \frac{BT}{OY} = \frac{\frac{\sqrt{3}}{2}}{r}$$

$$\frac{2\sqrt{3}}{\frac{3\sqrt{5}}{2} + r} = \frac{\frac{\sqrt{3}}{2}}{r} \implies 2r\sqrt{3} = \frac{3\sqrt{15}}{4} + \frac{r\sqrt{3}}{2}$$

Multiply both sides by $\frac{2}{\sqrt{3}}$:

$$4r = \frac{3\sqrt{5}}{2} + r$$

$$r = \frac{\sqrt{5}}{2} \to r^2 = \frac{5}{4} \to 5 + 4 = \boxed{9}$$

14. Let a_n denote the positive integer whose digits are n ones in base ten. For example, $a_3 = 111$. Let S be the sum of the digits of $(3a_{2025} + 1)^2$ and T be the sum of the digits of $(6a_{2025} + 1)^2$. Compute $S + T$.

Proposed by Yinuo Huang

Answer. 36452

Solution. Trying out some smaller numbers tells us $(3a_n+1)^2 = \underbrace{11\dots1}_{n \text{ times}}\underbrace{55\dots5}_{n-1 \text{ times}}6$ and $(6a_n+1)^2 = \underbrace{44\dots4}_{n \text{ times}}\underbrace{88\dots8}_{n-1 \text{ times}}9$. In fact, we can prove this. Then if we let $n = 2025$, $S = n+5(n-1)+6 = 6n+1$ and $T = 4n+8(n-1)+9 = 12n+1$, so $S+T = 18n+2 = 18\times 2025+2 = \boxed{36452}$.

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15. If x and y are positive real numbers such that

$$\frac{25}{x} + \frac{2025}{y} = 1,$$

compute the minimum possible value of $x+y$.

Proposed by Yinuo Huang

Answer. 2500

Solution. Let $25/x = \sin^2(a), 2025/y = \cos^2(a)$, so we have the given equation by the Pythagorean identity.

$$x+y = 25/\sin^2(a) + 2025/\cos^2(a) = 25\csc^2(a) + 2025\sec^2(a)$$
$$= 25(\cot^2(a)+1) + 2025(\tan^2(a)+1) = 2050 + 25\cot^2(a) + 2025\tan^2(a)$$

By the AM-GM inequality,

$$\geq 2050 + 2\sqrt{25\times 2025} = 2050 + 2\times 5\times 45 = \boxed{2500}$$

where equality is achieved when

$$25\cot^2(a) = 2025\tan^2(a) \implies \tan(a) = \pm\frac{1}{3} \implies y = 9x$$

where $x = 250, y = 2250$ and $x+y = \boxed{2500}$.

Also, there is a non-trig solution as well. Can you find it?

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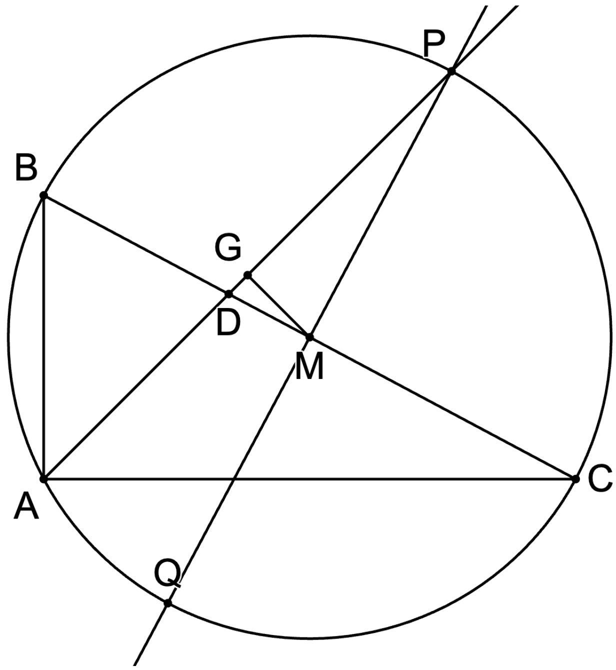
16. Consider a triangle $\triangle ABC$ with $BC = 17$, $CA = 15$, and $AB = 8$. Suppose the midpoint of BC is M and the interior bisector of $\angle A$ intersects the perpendicular bisector of BC at P. Find the area of $\triangle AMP$.

Proposed by Dennis Chen

Answer. $\dfrac{161}{8}$

Solution. We claim that P is on the circumcircle of $\triangle ABC$. The reason this is true is because of the following. Let MP intersect the circumcircle of ABC at X. Let the bisector of $\angle A$ intersect this circle at Y. Note that arcs BY and CY are congruent since they are subtended by congruent 45° angles. Also, segments BX and CX are congruent. Since X and Y are on the circle, this implies that $X = Y$. Thus, the intersection point P is on the circumcircle.



Knowing this, we can set up similar triangles. Let Q be the point opposite P on the circumcircle, and let D be the intersection of AP and BC. From the angle bisector theorem, $\frac{BD}{CD} = \frac{8}{15}$, and knowing that $BD = 17$, we can deduce that $BD = \frac{136}{23}$. Since $BM = \frac{17}{2}$, we can find that

$$DM = \frac{17}{2} - \frac{136}{23} = \frac{119}{46}$$

Notice that $\triangle AMP$ is isosceles. We just need to find the base and height to find the area. For $\triangle DMP$, $DM = \frac{119}{46} = \frac{7 \cdot 17}{46}$ and $MP = \frac{17}{2}$, so

$$DP = \sqrt{\left(\frac{7 \cdot 17}{46}\right)^2 + \left(\frac{17}{2}\right)^2} = \sqrt{\left(\frac{7 \cdot 17}{46}\right)^2 + \left(\frac{17 \cdot 23}{46}\right)^2} = \frac{17}{46}\sqrt{7^2 + 23^2} = \frac{17}{46} \cdot 17\sqrt{2}$$

If we drop an altitude from M to DP and call this point G, we can find MG:

$$MG = \frac{DM \cdot MP}{DP} = \frac{\frac{17}{2} \cdot \frac{119}{46}}{\frac{17}{46} \cdot 17\sqrt{2}} = \frac{7}{2\sqrt{2}}$$

We can also find GP using similar triangles:

$$\frac{GP}{MG} = \frac{MP}{DM} \implies GP = \frac{MG \cdot MP}{DM} = \frac{\frac{7}{2\sqrt{2}} \cdot \frac{17}{2}}{\frac{119}{46}} = \frac{23}{2\sqrt{2}}$$

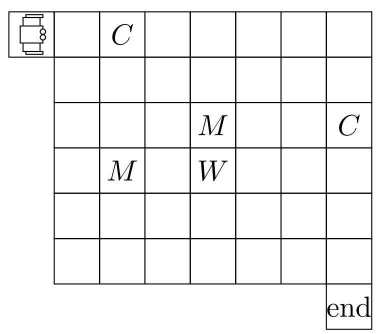
The area of $\triangle AMP$ is therefore

$$\frac{AP \cdot MG}{2} = \frac{2 \cdot GP \cdot MG}{2} = GP \cdot MG = \frac{23}{2\sqrt{2}} \cdot \frac{7}{2\sqrt{2}} = \boxed{\frac{161}{8}}$$

17. On a 6×7 grid, a robot starts at a cell immediately west of the northwest corner, facing east. The end cell is the one immediately south of the southeast corner. Natalie has programmed the robot to move at each cell in the following manner:

- if the current cell has no letter, move to the next cell in the current direction;
- if the current cell contains the letter C, write this letter down on the paper, face south, and move to the next cell to the south;
- if the current cell contains the letter M, write this letter down on the paper, face east, and move to the next cell to the east;
- if the current cell contains the letter W, write this letter down on the paper, face north, and move to the next cell to the north.

Natalie wants to place the five letters C, M, W, M, C on distinct cells. Compute the number of such placements that will cause the robot to eventually reach the end cell and, upon reaching it, to have written exactly the sequence C, M, W, M, C in that order. (Below is one example placement.)



Proposed by Yinuo Huang, Sreeram Vuppala, Nathan Ye, Avnith Vijayram

Answer. 225

Solution. Notice that in any case, the robot will move right, then down, then right, then up, then right, then down. Out of the seven columns, the robot uses three of them for vertical movement. The last column is always used, and from the remaining six, any two distinct columns are used, so we can have $\binom{6}{2} = 15$ different sets of columns to use for vertical motion.

As for the six rows, after the robot starts to move down, the robot uses two of these rows to continue its horizontal movement. Similar to the columns, there are $\binom{6}{2} = 15$ different sets of rows for the robot to use for horizontal movement. (Note that the robot cannot use the same row for both movements.)

We can then multiply the possible sets of rows and possible sets of columns to get an answer of $\binom{6}{2}\binom{6}{2} = 15 \cdot 15 = \boxed{225}$.

18. Let m and n be positive integers. If $\dfrac{n^5+m}{n-1}$ is also a positive integer, find the smallest m such that there are 12 solutions for n.

Proposed by BangTam Ngo

Answer. 59

Solution. First off, you realize that the fraction above simply needs to have an integer solution. You then can check to see how the nominator is divisible by the denominator.

$$n^5 + m \equiv 0 \mod n-1$$

Then you subract it by 1 since $n^5 - 1 = (n-1)(n^4+n^3+n^2+n+1)$

$$n^5 + m - 1 \equiv -1 \mod n-1$$
$$m \equiv -1 \mod n-1$$

To minimize m such that there are 12 solutions to n, we have to find a number with 12 factors as its $\mod n-1$ for any n, then subtract 1 to find m. All 12 factors of this certain m are all possible $n-1$.

The minimum number would be

$$2^2 * 3 * 5 = 60$$
$$m = 60 - 1 = \boxed{59}$$

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19. Given a triangle ABC, let $AB = 7$, $BC = 8$, and $CA = 9$. Let E be the point on circle ABC not equal to A such that $AE \perp BC$. Express $CE = \frac{m}{\sqrt{n}}$, where m and n are positive integers with n squarefree. Find $m+n$.

Proposed by Chris Lu

Answer. 19

Solution. Let E' be the antipode of A. Let O and H be the circumcenter and orthocenter of triangle ABC, respectively. Since O and H are isogonal conjugates, $\angle BAE = \angle CAE'$

So $BE = CE'$. This implies that $BCE'E$ is an isosceles trapezoid, so $CE = BE'$.

Since AE' is a diameter, we know $AE' = 2R$.

By Heron's, $[ABC] = \sqrt{(12)(3)(4)(5)} = 12\sqrt{5}$. Then using the formula $[ABC] = \frac{abc}{4R}$, we get

$$R = \frac{abc}{4[ABC]} = \frac{(7)(8)(9)}{48\sqrt{5}} = \frac{21}{2\sqrt{5}}$$

So $AE' = \frac{21}{\sqrt{5}}$.

Now by the Pythagorean Theorem, we have

$$BE' = \sqrt{AE'^2 - AB^2} = \sqrt{\left(\frac{21}{\sqrt{5}}\right)^2 - 7^2} = \frac{14}{\sqrt{5}}$$

Therefore,

$$CE = \frac{14}{\sqrt{5}}$$

$m = 14$ and $n = 5$, so the answer is $\boxed{19}$.

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20. Denote $\overline{z}$ as the complex conjugate of z. Let z_1 and z_2 be complex numbers such that

$$(z_1\overline{z_2}) + (\overline{z_1}z_2) = 32$$

and

$$|z_1\overline{z_2}| \cdot |\overline{z_1}z_2| = 300.$$

If z_1 and z_2 represent points A and B on the complex plane with origin O, compute the area of $\triangle AOB$.

Proposed by Yinuo Huang

Answer. $\sqrt{11}$

Solution. The second condition simplifies to $|z_1|^2|z_2|^2 = 300$, so $|z_1||z_2| = 10\sqrt{3}$. Also, suppose that $z_1 = a + bi$ and $z_2 = c + di$, then the first condition simplifies to $ac + bd = 16$. Let θ be the angle between vectors (a, b) and (c, d). Then, $(a, b) \cdot (c, d) = |z_1||z_2| \cos\theta = 16$. Then $\cos\theta = \frac{8\sqrt{3}}{15}$, and because we consider θ as an angle in a triangle, $\sin\theta > 0$, so $\sin\theta = \sqrt{1 - \cos^2\theta} = \frac{\sqrt{33}}{15}$. Then $A = \frac{1}{2}|z_1||z_2| \sin\theta = \frac{1}{2} \cdot 10\sqrt{3} \cdot \frac{\sqrt{33}}{15} = \boxed{\sqrt{11}}$.

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21. Compute the sum of all positive integers m such that

$$\sum_{k=1}^{1000} \left\lfloor \frac{km}{1000} \right\rfloor = 500m.$$

Proposed by Yinuo Huang

Answer. 13660

Solution. Let $n = 1000$ and define

$$S = \sum_{k=1}^{n-1} \left\lfloor \frac{km}{n} \right\rfloor.$$

Then

$$2S = \sum_{k=1}^{n-1} \left\lfloor \frac{km}{n} \right\rfloor + \left\lfloor \frac{(n-k)m}{n} \right\rfloor = \sum_{k=1}^{n-1} \begin{cases} m, & n \mid km, \\ m-1, & n \nmid km, \end{cases}$$

so

$$2S = (\gcd(m,n)-1)m + \big((n-1) - (\gcd(m,n)-1)\big)(m-1) = \gcd(m,n) + mn - n - m,$$

and hence

$$S = \frac{\gcd(m,n) + mn - n - m}{2}.$$

Since we are given that $S + \left\lfloor \frac{nm}{m} \right\rfloor = \frac{nm}{2}$, it follows that $S = \frac{mn}{2} - m$. Therefore,

$$\frac{\gcd(m,n) + mn - n - m}{2} = \frac{mn}{2} - m \quad \Longrightarrow \quad \gcd(m,n) + mn - n - m = mn - 2m$$

and thus

$$\gcd(m,n) = n - m.$$

By the Euclidean algorithm,

$$\gcd(m,n) = n - m \quad \Longrightarrow \quad \gcd(n-m,n) = n - m.$$

Let $d = n - m$. Then $\gcd(d,n) = d$, so $d \mid n$. Since $m \geq 1$, we have $1 \leq d \leq n-1$. Hence

$$\sum m = \sum (n-d) = \sum n - \sum d = n\big(\sigma_0(n) - 1\big) - \big(\sigma_1(n) - n\big) = n\,\sigma_0(n) - \sigma_1(n).$$

For $n = 1000 = 2^3 \cdot 5^3$, we have $\sigma_0(1000) = (3+1)(3+1) = 16$ and

$$\sigma_1(1000) = (1+2+4+8)(1+5+25+125) = 2340.$$

Therefore,

$$n\,\sigma_0(n) - \sigma_1(n) = 1000 \cdot 16 - 2340 = \boxed{13660}.$$