

CMWMC 2023, Guts Round, Set 1/8

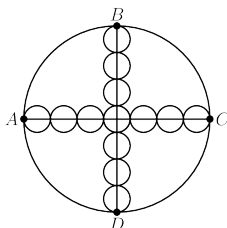
1. Sherry starts with a three-digit positive integer. She subtracts 7 from it, then multiplies the result by 7, and then adds 7 to that. If she ends up with 2023, what number did she start with?
2. Square $ABCD$ has side length 1. Point X lies on $\overline{AB}$ such that $\frac{AX}{XB} = 2$, and point Y lies on $\overline{DX}$ such that $\frac{DY}{YX} = 3$. Compute the area of triangle DAY .
3. A fair six-sided die is labeled 1-6 such that opposite faces sum to 7. The die is rolled, but before you can look at the outcome, the die gets tipped over to an adjacent face. If the new face shows a 4, what is the probability the original roll was a 1?

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4. What is $\gcd(2^6 - 1, 2^9 - 1)$?
5. Sarah is walking along a sidewalk at a leisurely speed of $\frac{1}{2}$ m/s. Annie is some distance behind her, walking in the same direction at a faster speed of s m/s. What is the minimum value of s such that Sarah and Annie spend no more than one second within one meter of each other?
6. You have a choice to play one of two games. In both games, a coin is flipped four times. In game 1, if (at least) two flips land heads, you win. In game 2, if (at least) two *consecutive* flips land heads, you win. Let N be the number of the game that gives you a better chance of winning, and let p be the absolute difference in the probabilities of winning each game. Find $N + p$.

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7. Let A, B, C , and D be equally spaced points on a circle O . 13 circles of equal radius lie inside O in the configuration below, where all centers lie on $\overline{AC}$ or $\overline{BD}$, adjacent circles are externally tangent, and the outer circles are internally tangent to O . Find the ratio of the area of the region inside O but outside the smaller circles to the total area of the smaller circles.



8. Find the greatest divisor of $40!$ that has exactly three divisors.
9. Suppose we have positive integers a, b, c such that $a = 30$, $\text{lcm}(a, b) = 210$, $\text{lcm}(b, c) = 126$. What is the minimum value of $\text{lcm}(a, c)$?

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10. Square $ABCD$ has side length $n > 1$. Points E and F lie on $\overline{AB}$ and $\overline{BC}$ such that $AE = BF = 1$. Suppose $\overline{DE}$ and $\overline{AF}$ intersect at X and $\frac{AX}{XF} = \frac{11}{111}$. What is n ?
11. Let x be the positive root of $x^2 - 10x - 10 = 0$. Compute $\frac{1}{20}x^4 - 6x^2 - 45$.
12. Francesca has 7 identical marbles and 5 distinctly labeled pots. How many ways are there for her to distribute at least one (but not necessarily all) of the marbles into the pots such that at most two pots are nonempty?



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13. Suppose $\overline{AB}$ is a radius of a circle. If a point C is chosen uniformly at random inside the circle, what is the probability that triangle ABC has an obtuse angle?
14. Find the second smallest positive integer c such that there exist positive integers a and b satisfying the following conditions:
- $5a = b = \frac{c}{5} + 6$.
 - $a + b + c$ is a perfect square.
15. A spotted lanternfly is at point $(0, 0, 0)$, and it wants to land on an unassuming CMU student at point $(2, 3, 4)$. It can move one unit at a time in either the $+x$, $+y$, or $+z$ directions. However, there is another student waiting at $(1, 2, 3)$ who will stomp on the lanternfly if it passes through that point. How many paths can the lanternfly take to reach its target without getting stomped?

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16. Let $P(x)$ be a quadratic such that $P(-2) = 10, P(0) = 5, P(3) = 0$. Then, find the sum of the coefficients of the polynomial equal to $P(x)P(-x)$.
17. Suppose that $a < b < c < d$ are positive integers such that the pairwise differences of a, b, c, d are all distinct, and $a + b + c + d$ is divisible by 2023. Find the least possible value of d .
18. Consider a right rectangular prism with bases $ABCD$ and $A'B'C'D'$ and other edges $\overline{AA'}$, $\overline{BB'}$, $\overline{CC'}$ and $\overline{DD'}$. Suppose $AB = 1$, $AD = 2$, and $AA' = 1$.
- Let X be the plane passing through A , C' , and the midpoint of $\overline{BB'}$.
 - Let Y be the plane passing through D , B' , and the midpoint of $\overline{CC'}$.

Then the intersection of X , Y , and the prism is a line segment of length ℓ . Find ℓ .

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19. Sequences a_n and b_n of positive integers satisfy the following properties:

- (1) $a_1 = b_1 = 1$
- (2) $a_5 = 6, b_5 \geq 7$
- (3) Both sequences are strictly increasing
- (4) In each sequence, the difference between consecutive terms is either 1 or 2
- (5) $\sum_{n=1}^5 na_n = \sum_{n=1}^5 nb_n = S$

Compute S .

20. Let A , B , and C be points lying on a line in that order such that $AB = 4$ and $BC = 2$. Let $\mathcal{I}$ be the circle centered at B passing through C , and let D and E be distinct points on $\mathcal{I}$ such that AD and AE are tangent to $\mathcal{I}$. Let $\mathcal{J}$ be the circle centered at C passing through D , and let F and G be distinct points on $\mathcal{J}$ such that AF and AG are tangent to $\mathcal{J}$ and $DG < DF$. Compute the area of quadrilateral $DEFG$.
21. Twain is walking randomly on a number line. They start at 0, and flip a fair coin 10 times. Every time the coin lands heads, they increase their position by 1, and every time the coin lands tails, they decrease their position by 1. What is the probability that at some point the absolute value of their position is at least 3?
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22. Find the unique ordered pair (m, n) of positive integers such that $x = \sqrt[3]{m} - \sqrt[3]{n}$ satisfies $x^6 + 4x^3 - 36x^2 + 4 = 0$.
23. Jenny plays with a die by placing it flat on the ground and rolling it along any edge for each step. Initially the face with 1 pip is face up. How many ways are there to roll the dice for 6 steps and end with the 1 face up again?
24. There exists a unique positive five-digit integer with all odd digits that is divisible by 5^5 . Find this integer.
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