Team Round Solutions

1. Equiangular hexagon $NARUTO$ has sides $NA = 5$, $AR = 4$, $RU = 1$, $UT = 7$, $TO = 2$, and $ON = 3$. If the area of $NARUTO$ can be expressed as $\frac{a\sqrt{b}}{c}$ in simplest form, find the value of $a + b + c$.

   Proposed by BangTam Ngo

   Answer. 40

   Solution. $NARUTO$ can be constructed by taking an equilateral triangle with side length 10 and cutting equilateral triangles of side lengths 1, 2, and 5 from the corners. Thus $NARUTO$ has an area of $\frac{\sqrt{3}}{4} \cdot (100 - 25 - 4 - 1) = \frac{70\sqrt{3}}{4} = \frac{35\sqrt{3}}{2}$. $35 + 3 + 2 = \boxed{40}$

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2. Jenny draws a 2×20 lattice. She picks one lattice point to be red and a different lattice point to be blue. And then she challenges Sam to draw a path from red to blue, where each segment of the path is 1 unit long and in cardinal directions, and going through each lattice point exactly once (excluding the start and end points). For how many choices of red and blue lattice points is it possible for Sam to succeed.

   Proposed by Alan Abraham

   Answer. 764

   Solution. A parity argument shows that the taxicab distance between the red and blue points must be odd. This is because each segment of the path toggles the parity of the taxicab distance from the start and we take $2(20) - 1$, an odd number, of steps to create the path.

   One can observe that it for almost every pair of possible red and blue points that satisfy the above condition, we can construct such a path. The only time it is not possible is when the points are in the same column and not at the corner of the lattice. One can show that it is not possible to construct such a path. Hence, the total is $4 + 2(20)(20 - 1) = \boxed{764}$

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3. James runs a business making customized helmets, and recently received a commission from Michael. Michael's head shape is quite strange however, so he sent James a front, side, and top down view of his head. In each of the three images, Michael's head appears to be a circle of radius 1. Given that James only makes spherical helmets, he is wondering: What is the minimum R such that Michael's head is guaranteed to fit in a sphere of radius R?

   Proposed by James Yang

   Answer. $\frac{\sqrt{6}}{2}$

   Solution. Michael's head is contained in the set of points in 3d space characterized by the inequalities $x^2 + y^2 \leq 1$, $x^2 + z^2 \leq 1$, and $y^2 + z^2 \leq 1$. Then the furthest possible distance a point can be from the origin is $x^2 + y^2 + z^2 \leq \frac{3}{2}$ by adding the inequalities. This is tight since we can check $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ lies in the set. Since this set is also symmetric around the origin, it's clear that the answer is just $\sqrt{\frac{3}{2}} = \boxed{\frac{\sqrt{6}}{2}}$.

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4. Find the greatest ϵ such that $f(n) = \frac{\sigma(n)}{n^\epsilon}$ is maximized at some n divisible by 2026, where $\sigma(n)$ is the number of positive factors of n.

   Answer. $\log_{1013} 2$

   Proposed by Maxwell Du

   Solution. Given ϵ, we can greedily find the optimal n. We will start at 1. For each prime p, if its exponent in the current number's prime factorization is a, then $\frac{\sigma(np)}{(np)^\epsilon} = \frac{(a+2)/(a+1)}{p^\epsilon}\frac{\sigma(n)}{n^\epsilon}$. Thus, $f(n)$ does not decrease when multiplying n by p if $\frac{(a+2)/(a+1)}{p^\epsilon} \geq 1$. The PF of 2026 is $2 \cdot 1013$, so to be able to multiply by 2 and 1013, we need to have $\frac{2}{1013^\epsilon} \geq 1$, giving $1013^\epsilon \leq 2$.

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5. There is a unique nonzero complex number z such that the system of equations $|w-8| = 5$, $|w+2+5i| = 5\sqrt{2}$, $|w| = |w-z|$ has more than one solution in w. Find z.

   Proposed by John Li

   Answer. $\frac{12\sqrt{5}}{5}$

   Solution. Note that the first two equations represent circles in the complex plane, while the final equation represents a line. In particular, it is the perpendicular bisector of the segment between z and the origin. Since we want the whole system to have more than one solution, we are looking for the line that passes through the two intersection points of the circles. In Cartesian coordinates, the equations of the circles are $(x-8)^2 + y^2 = 25$ and $(x+2)^2 + (y+5)^2 = 50$. If we expand and subtract the second equation from the first, we get $20x + 10y - 35 = 25$, which simplifies to $2x + y = 6$. Note that any simultaneous solution to the original equations must also satisfy this one. Since this is an equation of a line, this is precisely the line passing through both intersection points, which was our goal. Now we must find the z such that $|w| = |w-z|$ represents the line $2x + y = 6$. Since the equation represents the perpendicular bisector of the segment between z and the origin, we must reflect the origin over $2x + y = 6$. This gives us the point $\left(\frac{24}{5}, \frac{12}{5}\right)$, so we have $z = \frac{24}{5} + \frac{12}{5}i = \frac{12}{5}(2+i)$, so $|z| = \frac{12}{5}|2+i| = \boxed{\frac{12\sqrt{5}}{5}}$.

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6. The superfactorial of a positive integer n is $F(n) = 1! \cdot 2! \cdot 3! \cdot \cdots \cdot n!$.

   Find the product of all positive integers $n \leq 20$ such that there exists a positive integer m where $\frac{F(n)}{m!}$ and $\frac{F(n)}{(m+1)!}$ are both perfect squares.

   Proposed by Justin Hsieh

Answer. 1792

Solution. First, a must be one less than a perfect square, since dividing by $a+1$ preserves the perfect square. Next, observe that

$$\frac{F(x)}{x!!} = \frac{x!(x-1)!}{x} \cdot \frac{(x-2)!(x-3)!}{x-2} \cdot \frac{(x-4)!(x-5)!}{x-4} \cdots = ((x-1)!)^2 \cdot ((x-3)!)^2 \cdot ((x-5)!)^2 \cdots$$

is a perfect square, where $x!! = x(x-2)(x-4)\cdots$ is the double factorial. Therefore it suffices to find a such that $\frac{x!!}{a!}$ is a perfect square.

If x is even, we have

$$x!! = x \cdot (x-2) \cdot (x-4) \cdots 4 \cdot 2 = 2(\tfrac{x}{2}) \cdot 2(\tfrac{x}{2}-1) \cdots 2(\tfrac{x}{2}-2) \cdots 2(2) \cdot 2(1) = 2^{x/2}(\tfrac{x}{2})!$$

We can show that $x = 8, 14, 16$ work (with $a = 3, 8, 8$, respectively). Nothing else works by analyzing the square-free part. Multiplying the three together yields:

$$8 \times 14 \times 16 = \boxed{1792}$$

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7. An 8-sided die is crafted, and the probability p_i of the i-th face being rolled is generated as follows: p_1 is a uniformly random number from 0 to 1, p_k is a uniformly random number from 0 to $1 - \sum_{i=1}^{k-1} p_i)$ for all k from 2 to 7, with the p_k generated iteratively in order. The 8th side has all the leftover probability; $p_8 = 1 - \sum_{i=1}^{7} p_i$. This die is then rolled 8 times. Find the probability that in these 8 rolls, each face from 1 to 8 shows up exactly once.

Proposed by Hari Desikan

Answer. $\frac{2}{9!} = \frac{1}{181440}$

Solution. We will split this up as a conditional probability problem. The solution will be that the probability of one occurrence for each value is the product of k showing up exactly once given all $i < k$ shows up once for each k. We will compute each of these directly.

Suppose that for some given $k < 8$, each of the first $k-1$ have shown heads exactly once (this works for $k = 1$ as well). Then, there are exactly $9-k$ dice that are from the values $k, \ldots, 8$. Now, for any dice that isn't $9-k$, the probability that it is k is some uniformly random number form 0 to $1 - \sum_{i=1}^{k-1} p_i$. Conditioned on the fact that it isn't 1 to $k-1$, the probability is uniformly random between 0 and 1.

Then, we compute the probability that if the conditional probability of showing up as k is between 0 and 1, then exactly one of the remaining $9-k$ coins is k. We do this by doing the following: Pick some random number from 0 to 1, and let this be the probability p. Then, for the $9-k$ coin flips, choose the coin flip by picking a random number from 0 to 1, and making it heads if it is less than p and tails otherwise. We have exactly one head if there is exactly one of these random drawings less than p, which is the same as if our value p is the second largest of the $10-k$ uniform drawings. This happens with the probability of $\frac{1}{10-k}$.

If $k = 8$, then the conditional probability is 8. Thus, the total answer is $\frac{1}{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3} = \frac{2}{9!} = \frac{1}{181440}$

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8. The products in an $n \times n$ multiplication table are summed up to obtain S. Find the smallest $n > 1$ such that S has exactly n factors.

Proposed by Emir Naduvil Valappil

Answer. 45

Solution. One can observe that the sum of the products in an $n \times n$ multiplication table is equivalent to $(1 + 2 + ...n)(1 + 2 + ...n) = (\frac{n(n+1)}{2})(\frac{n(n+1)}{2}) = (\frac{n(n+1)}{2})^2$. Because the sum is a perfect square, it follows that the number of factors of S is odd, which implies that n is odd. We can see that n cannot prime because regardless of n we know that for $n > 1$ that $\frac{n(n+1)}{2}$ is composite, so there will be more factors in $(\frac{n(n+1)}{2})^2$ than n^2. Hence it suffices to test non-prime and odd $n > 1$. Furthermore, upon inspection, we see that $n = a \cdot b$ where a, b are distinct odd primes does not work either for the same reason: $\frac{n(n+1)}{2}$ would further introduce more factors which would guarantee that n is not equal to the sum of factors of n. Thus our lowest qualifying cases gives us $n = 27, 45, 63....$ We find that $n = 27$ does not work but $n = 45$ gives $S = (1035)^2 = (3^2 \cdot 5 \cdot 23)^2 = 3^4 \cdot 5^2 \cdot 23^2$ which has $(4+1)(2+1)(2+1) = 5 \cdot 3 \cdot 3 = 45 = n$, so our lowest solution is $\boxed{n = 45}$

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9. Let $\triangle ABC$ be a right triangle with $AB = 33, BC = 44, AC = 55$. Let X, Y be on the sides AB, BC respectively such that $AX = 11, CY = 11$. Let AY intersect CX at P. Then, let E, F be on AX, CY such that $XE = CF$. Then, let EF intersect AY, CX at the points R and S respectively. Find the minimum circumradius of the triangle $\triangle PRS$.

Proposed by Henry Zheng

Answer. $\frac{11\sqrt{5}}{2}$

Solution. Let the circumcircle of AXP, CYP intersect each other at P and at O, I claim that O always lies on the circumcircle of $\triangle PRS$. Notice that by the properties of spiral similarity, consider the way that O maps AX to CY, and the point E to F, along that same rotation/spiral similarity. The spiral similarity is unique, and since the ratio is one, it's just a rotation. However, since O sends AX, AE, EX to YC, YF, FC respectively, we get that the circumcircles of $\triangle AXP, \triangle CYP, \triangle AER, \triangle YFR, \triangle CFS, \triangle XES$ all intersect at O. Then, by cyclic quadrilaterals, we can note that:

$$\angle AOP = 180 - \angle AXP, \angle AOR = 180 - \angle AER$$

Therefore, this gets us that:

$$\angle ESX = \angle AER - \angle AXP = \angle RSP = \angle ROP$$

Thus, we get that O, P, R, S are cyclic. Therefore, since OP is always a chord in the circumcircle of $\triangle PQR$, the maximum circumradius is $OP/2$. Let's coordinate bash this. Setting $A = (33, 0), B =$

$(0, 44), X = (22, 0), Y = (0, 33)$. Then, the intersection of AY and CX ends up to be $P = (11, 22)$. Then, finding the circumcenter, we get that the circumcenter of $\triangle AXP$ is $O_A = (27.5, 16.5)$, and the circumcenter of $\triangle CYP$ is $O_C = (16.5, 38.5)$. Then, the radii for each of the circumcircles is the same, and is equal to:

$$\frac{11\sqrt{10}}{2} = r$$

Then, the distance between the two centers is $11\sqrt{5}$. Then, we have that the distance between PQ is equal to:

$$\sqrt{(11\sqrt{10})^2 - (11\sqrt{5})^2} = 11\sqrt{5}$$

Dividing that by two yields the final answer of:

$$\boxed{\frac{11\sqrt{5}}{2}}$$

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10. Suppose $P(x)$ is a polynomial of degree 2024 such that for all $0 \le n \le 2024$, we have $P(n) = F_n$. (Here, F_n is the Fibonacci sequence given by $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \ge 2$.) Find $P(2025)$.

Proposed by Alan Abraham

Answer. 0

Solution. We know from binet's formula that $F_n = \frac{\phi^n - \varphi^n}{\sqrt{5}}$ where $\phi = \frac{1+\sqrt{5}}{2}$ and $\varphi = \frac{1-\sqrt{5}}{2}$. So for $0 \le n \le 2024$, $P(n) = \frac{1}{\sqrt{5}}\phi^n - \frac{1}{\sqrt{5}}\varphi^n$. Using binomial theorem, we can rewrite the RHS as

$$\frac{1}{\sqrt{5}}\sum_{k=0}^{n}\binom{n}{k}(\phi-1)^k - \frac{1}{\sqrt{5}}\sum_{k=0}^{n}\binom{n}{k}(\varphi-1)^k$$

Note that for $k > n$ we have $\binom{n}{k} = 0$. So this means for $n \le 2024$ we can rewrite this as

$$\frac{1}{\sqrt{5}}\sum_{k=0}^{2024}\binom{n}{k}(\phi-1)^k - \frac{1}{\sqrt{5}}\sum_{k=0}^{2024}\binom{n}{k}(\varphi-1)^k$$

We note that $\binom{n}{k}$ is a polynomial in n of degree k, so this entire expression is a polynomial in n of degree 2024. Since $P(n)$ and this polynomial agree on 2025 different inputs, they must be the same, so

$$P(n) = \frac{1}{\sqrt{5}}\sum_{k=0}^{2024}\binom{n}{k}(\phi-1)^k - \frac{1}{\sqrt{5}}\sum_{k=0}^{2024}\binom{n}{k}(\varphi-1)^k$$

Before we compute $P(2025)$ we can make some simplications. Note that $\phi + \varphi = 1$, so $P(n)$ can be rewritten as

$$P(n) = \frac{1}{\sqrt{5}}\sum_{k=0}^{2024}\binom{n}{k}(-\varphi)^k - \frac{1}{\sqrt{5}}\sum_{k=0}^{2024}\binom{n}{k}(-\phi)^k$$

$$P(n) = \sum_{k=0}^{2024} \binom{n}{k} (-1)^{k+1} F_k$$

Note that we can see that this means $F_{2025} = \sum_{k=0}^{2024} \binom{n}{k} (-1)^{k+1} F_k$. We have

$$P(2025) = \sum_{k=0}^{2024} \binom{2025}{k} (-1)^{k+1} F_k$$

$$P(2025) = -\binom{2025}{2025} (-1)^{2025+1} F_{2025} + \sum_{k=0}^{2025} \binom{2025}{k} (-1)^{k+1} F_k$$

$$P(2025) = -F_{2025} + F_{2025}$$

$$P(2025) = \boxed{0}$$

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11. **(Tiebreaker)** Let $F_{1000000}$ be the one millionth Fibonacci number in base 10. Let $f(n)$ for a positive integer n denote how many maximal strictly increasing subsequences we can partition n into. For example,

$$f(135246) = 2$$

because there are two maximal increasing sequences, 135 and 246. Another example to consider is:

$$f(74382) = 4$$

because there are 4 maximal increasing sequences, $7, 4, 38, 2$. Estimate the value of:

$$f(F_{1000000})$$

Proposed by Henry Zheng

Answer. 114914

Solution. Just doing this through some program, we get the value to be 114914.