Geometry Round Solutions

1. Consider triangle $\triangle ABC$ with side lengths 6, 7, and 10. Sebastian places a point P somewhere in the plane and calculates $\max(AP, BP, CP)$. What is the minimum possible number he could have calculated?

   Proposed by Sebastian Kumar

   Answer. 5

   Solution. Notice that $\triangle ABC$ is an obtuse triangle, as $6^2 + 7^2 < 10^2$. Therefore, if we choose any point P, then by triangle inequality $\overline{PA} + \overline{PC} \geq 10$ which means that $\max(AP, BP, CP) \geq 5$. Now, we can note that since the triangle is obtuse, the circumcircle with the longest side as its diameter must contain the triangle, meaning that therefore choosing P as the midpoint does ensure that $PB \leq 5$. Therefore, $\boxed{5}$ is our final answer.

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2. Two spheres are externally tangent to each other, one with radius 7 and one with radius 5. What is the side length of the smallest cube that fully contains both?

   Proposed by John Li

   Answer. $12 + 4\sqrt{3}$

   Solution. The optimal configuration is where the two spheres are aligned along a diagonal of the cube, so that each sphere is tangent to three sides of the cube. Now that we know what the diagram looks like, we can compute the side length of the cube. Let the center of the radius 7 cube be C_1, and the center of the radius 5 cube be C_2. Note that the segment $\overline{C_1C_2}$ has length $5+7 = 12$. Because of the orientation of the spheres, this segment must also be pointing along the diagonal of the cube. This means that $\overline{C_1C_2}$ covers a length of $\frac{12}{\sqrt{3}} = 4\sqrt{3}$ along each axis of the cube. We can now compute the side length by adding together the horizontal distances from one face to C_1, from C_1 to C_2, and from C_2 to the opposite face, giving us $7 + 4\sqrt{3} + 5 = \boxed{12 + 4\sqrt{3}}$.

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3. Non-degenerate triangle ABC has side lengths $n, n+1, n+2$, where n is an integer. Interestingly, the median altitude of this triangle is also an integer. Find the sum of the three lowest possible n.

   Proposed by Emir Naduvil Valappil

   Answer. 67

   Solution. One can compute the semiperimeter to be $s = \frac{a+b+c}{2} = \frac{n+n+1+n+2}{2} = \frac{3n+3}{2}$. Using Heron's formula, we have that the area $A = \sqrt{s(s-a)(s-b)(s-c)}$ which we can re-write as

$$(n+1) \cdot \frac{h_1}{2} = \sqrt{\frac{3n+3}{2} \cdot \frac{n+3}{2} \cdot \frac{n+1}{2} \cdot \frac{n-1}{2}}$$

where h_1 is the median altitude. The right side can simplify to $\frac{n+1}{4} \cdot \sqrt{3(n+3)(n-1)}$ so we now have that

$$(n+1) \cdot \frac{h_1}{2} = \frac{n+1}{4} \cdot \sqrt{3(n+3)(n-1)}$$

Solving for h_1 gives us

$$h_1 = \frac{\sqrt{3(n+3)(n-1)}}{2}$$

so now we seek to find when h_1 is an integer. If h_1 is an integer, then $\sqrt{3(n+3)(n-1)}$ is an even number; however, this directly implies that n cannot be even since the product of three odd numbers cannot be even. Then n is odd, and we can let $n-1 = 2b$ for some integer b The equation then simplifies to $\frac{\sqrt{3(2b+4)(2b)}}{2} = h_1 \implies \sqrt{3(b+2)(b)}$ is an integer Using pell equations or just bashing one can find the lowest 3 values of b to be $b = 1, 6, 25$ corresponding to $n = 3, 13, 51$ and their respective triangles: $3-4-5, 13-14-15, 51-52-53$, and the sum of these n is $3 + 13 + 51 = \boxed{67}$

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4. A cone has a circular base with a radius of 4. Its vertex is 6 units directly above the base, but it is offset from the center axis by 2 units. As the vertex rotates in a full circle around that center point (maintaining its height of 6), the slanted cone sweeps through space. What is the total volume of the solid formed by this motion?

Proposed by Lohith Tummala

Answer. $\frac{160\pi}{3}$

Solution. Let's calculate the volume of the big cone that the entire shape is inscribed in. The total height is 11, since we're offset from the central axis by two, which is one half of the radius. Therefore, the total height is $2 \cdot 6 = 12$. The radius is 4. Therefore, the volume of the big cone is:

$$\frac{h}{3}\pi r^2 = \frac{16 \cdot 12}{3}\pi = 64\pi$$

Now, the truncated cone is the bottom half of that, so therefore the total volume of the big cone minus the top half of the cone is $64\pi \cdot 7/8 = 56\pi$. Then, the volume of the "dented" cone in the middle is radius two. However, the height is $2/(4+2) \cdot 6 = 2$, by taking an axial cross section and doing similar triangles. Therefore, the volume of the dent is:

$$\frac{8\pi}{3}$$

Subtracting one from the other yields:

$$\boxed{\frac{160\pi}{3}}$$

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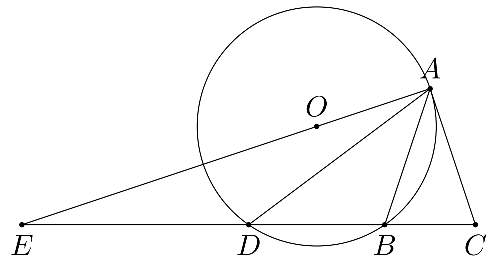
5. Let $\triangle ABC$ be an isosceles triangle with $AB = AC$. Let $D \neq B$ be the point on line $\overline{BC}$ such that the circle passing through A, B, and D is tangent to line $\overline{AC}$ at A. Let E be the point on line $\overline{BC}$ such that $AE \perp AC$. Given that $CD = 30$ and $AE = 3AC$, compute the area of $\triangle ADE$.

Proposed by Yinuo Huang

Answer. 270

Solution.



By the Tangent-Chord Theorem, $\angle ADB = \angle BAC$. Therefore, $\triangle ABC \sim \triangle DAC$, so $AD = DC = 30$. Let O denote the center of the circle through the points A, D, and B. Note that $OA \perp AC$, so A, O, E are collinear. Connect segment OB, and some angle chasing gives that $\angle AED = \angle EAD = \frac{1}{2}\angle BAC$, which means that $DE = AD = 30$ and $\angle ADE = 2\angle ABC$. Note that since $\frac{AE}{AC} = 3$, $\tan(\angle ACB) = \tan(\angle ABC) = 3$. Since both $\angle ABC$ and $\angle ACB$ must be acute, $\sin(\angle ABC) = \frac{3}{\sqrt{10}}$ and $\cos(\angle ABC) = \frac{1}{\sqrt{10}}$. Therefore, the area of $\triangle ADE$ is equal to $\frac{1}{2}(AD)(DE)(\sin(\angle ADE)) = \frac{1}{2}(30)(30)(2\sin(\angle ABC)\cos(\angle ABC)) = 900 \cdot \frac{3}{\sqrt{10}} \cdot \frac{1}{\sqrt{10}} = \boxed{270}$.

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6. Let $ABCD$ be a convex quadrilateral such that $AB = 8$, $BC = 27$, $CD = 36$. Let M and N be the midpoints of AC, BD respectively. Suppose that the angle bisectors of $\angle CBA, \angle DAB$ intersect at a point L, and that L lies on MN. Finally, suppose that $\cos(\angle CDA) = \frac{11}{23}$. Find the area of $\triangle LAB$.

Proposed by Henry Zheng

Answer. $\frac{36\sqrt{102}}{11}$

Solution. We will first prove the following claim.
Claim: If WXYZ is a quadrilateral with an inscribed circle with center I, then if P is the midpoint of WY, and Q is the midpoint of XZ, then I lies on PQ.

To prove this, it suffices to show that $[\triangle IPQ] = 0$. If this is true for $WXYZ$, then notice that if r is the inradius of the quadrilateral, then:

$$[\triangle IWX] + [\triangle IYZ] = 0.5r(\overline{WX} + \overline{YZ}) = 0.5r(\overline{XY} + \overline{WZ}) = [\triangle IXY] + [\triangle IWZ]$$

by the fact that there's an inscribed circle in it, which means that $WX + YZ = WZ + XY$. Now, we can also notice the following facts:

$$[\triangle PWX] = [\triangle PYX], [\triangle PWZ] = [\triangle PYZ]$$

$$[\triangle QWX] = [\triangle QWZ], [\triangle QYX] = [\triangle QYZ]$$

by the properties of medians. Therefore, we see that for $K = I, P, Q$, the following property holds:

$$[\triangle KWX] + [\triangle KYZ] = [\triangle KYX] + [\triangle KZW]$$

Since this is a relation based off of heights and area is a linear relationship in terms of heights (since the bases stay constant), these three points must lie on one line. An easy way to see this is that if we place the coordinates, then the heights $h(x, y)$ for a point (x, y) in the plane to each side is linear, meaning that the area is linear, meaning that a level set of the function of the difference of areas is also linear. Thus, we have finished proving the claim.

Now, since the angle bisectors of $\angle CBA, \angle DAB$ intersect on MN, let L be the center of a circle tangent to AB, BC, AD. Now, let the tangent line from D to the circle intersect BC at a point C'. Then, we know that since $ABC'D$ is a quadrilateral that has a circle inscribed in it with center L, then if M' is the midpoint of AC', then L lies on NM'. However, L also lies on NM. Therefore, either $L = N$, which would imply a rhombus, which doesn't make sense, or $M = M', C = C'$. Therefore, we see that $ABCD$ is actually a quadrilaterla with an inscribed circle. Therefore, we get that $AD = 36 + 8 - 27 = 17$. From here, by Law of Cosines, we have that:

$$36^2 + 17^2 - 2 \cdot 36 \cdot 17(11/23) = 8^2 + 27^2 - 2 \cdot 8 \cdot 27(\cos B)$$

$$792 - \frac{2 \cdot 36 \cdot 17}{23} = -2 \cdot (\cos B) \cdot 216$$

Dividing by 36 on both sides yields:

$$22 - \frac{34}{23} = -12 \cdot (\cos B)$$

$$\cos(B) = \frac{22 \cdot 23 - 34}{(-46 \cdot 12)} = \frac{-11}{23}$$

Therefore, we see that the quadrilateral is in fact cyclic, and we have that the semiperimeter is $8 + 36 = 44$. Therefore, by brahamgupta's formula, the area of the entire quadrilateral is:

$$\sqrt{8 \cdot 27 \cdot 36 \cdot 17} = 36\sqrt{17 \cdot 6}$$

Since $\triangle LAB$ is just $r(AB)/2$, this is about AB/p, where p is the perimeter of the entire shape. This comes out to be:

$$\frac{8}{88} \cdot 36\sqrt{102} = \boxed{\frac{36\sqrt{102}}{11}}$$

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7. Let M, B, C, A be points such that $MC = 21$. Let X, Y be points chosen such that X is on the same side of BC as M, Y is on the opposite side, MX is minimized, MY is maximized and that the following two angle relations hold:

$$\angle XBM - \angle XCA + \angle YBM + \angle YCA = 90^\circ$$

$$\angle YBM + \angle YCA - \angle XBM + \angle XCA + 360^\circ = 2(\angle BMC + \angle MCA)$$

Let $MX = 9$. It is known that $\angle CYA = \angle XCM = 1/2\angle CAX$ and that $AX = AY$. Find the area of $\triangle MCY$.

Proposed by Henry Zheng

Answer. $\frac{10290}{29}$

Solution. We will do the entire calculation in degrees, not radians. First we can separate the equations to get the following:

$$\angle XBM - \angle XCA = 225 - \angle BMC - \angle MCA$$

$$\angle YBM + \angle YCA = \angle BMC + \angle MCA - 135$$

Angle chasing again, yields us the following facts:

$$\angle BXC = 180 - \angle XCA + \angle XBM + \angle BMC = \angle BMC + 45$$

Similarly, we get that:

$$\angle BYC = 135 - \angle BMC$$

Therefore, this leads us to the fact that X, Y, B, C are cyclic. Since $AX = AB = AY$, we know that their circumcenter is A. Thus, since X, Y minimize and maximize MX, MY, it must be that they both lie on MA, which is the diameter that goes through M, with X closer to M, and Y farther from M. Then, notice again by angle chasing that MC is tangent to the circle. Therefore, since $MX = 9$, by power of a point, we get the radius to be 20. Thus, $AC = AB = 20$, and $\angle ACM = 90°$. Thus, $\triangle AMC$ is a $20 - 21 - 29$ right triangle. Then, since $AY = 20$, the area of

$$\triangle MYC = \frac{MY}{MA} \cdot [\triangle AMC] = \frac{49}{29} \cdot 210 = \boxed{\frac{10290}{29}}$$

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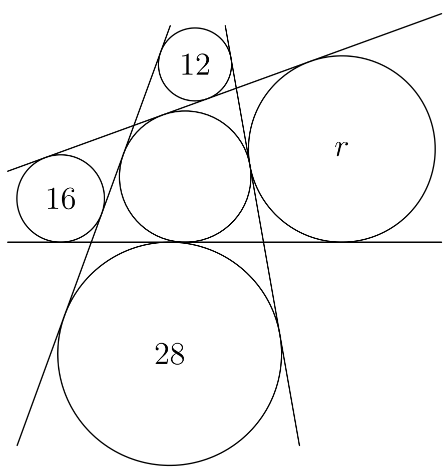
8. In the diagram below, four of the five drawn circles have radii marked as 12, 16, 28, and r. Find r.

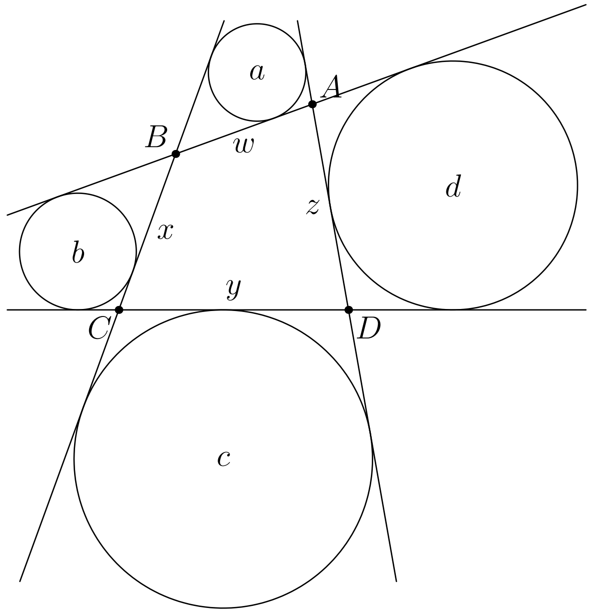


(Circles in the diagram are tangent to the lines, but they are not necessarily tangent to each other, even if they seem to be.)

Proposed by Dennis Chen

Answer. 21

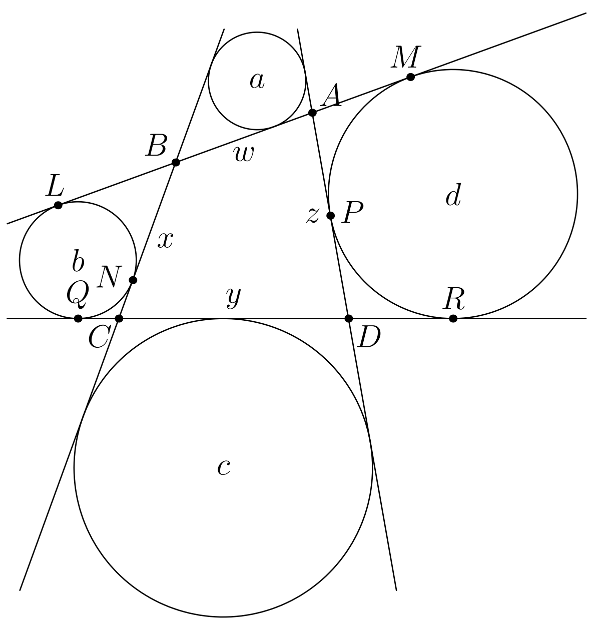
Solution. Since there are so many tangent lines in the diagram, length chasing seems like a promising line of attack. This motivates us to label the diagram as shown below.



Denote the area of $ABCD$ as K and the semiperimeter of $ABCD$ as s. (Note that $s = \frac{w+x+y+z}{2}$.) We make two claims about quadrilateral $ABCD$, *neither of which depend on the existence of its incircle* (this is why we erase it from the diagram), then synthesize them using $w + y = x + z = s$.

Claim 1: $K = a(s - w) + c(s - y) = b(s - x) + d(s - z)$.

Note that we just need to prove $K = a(s - w) + c(s - y)$ by symmetry. To do this, we mark a couple of tangency points in order to length chase.



We claim that $LM = QR = s$. Note that $LM = LB + BA + AM = NB + BA + AP$ and $QR = QC + CD + DR = CN + CD + DP$, so $LM + QR = 2s$. Since LM and QR are both common external tangents to the two circles, $LM = QR$, yielding our desired conclusion.

Now let O_B and O_D be the centers of the circles with radius b and d, respectively. Note that

$$K = [O_BO_DRQ] + [O_BO_DML] - [O_BLBN] - [O_BNCQ] - [O_DRDP] - [O_DPAM].$$

Since O_BO_DRQ is a trapezoid with bases O_BQ and O_DR, its area is

$$\frac{1}{2}(O_BQ + O_DR)QR = \frac{1}{2}(a+c)s.$$

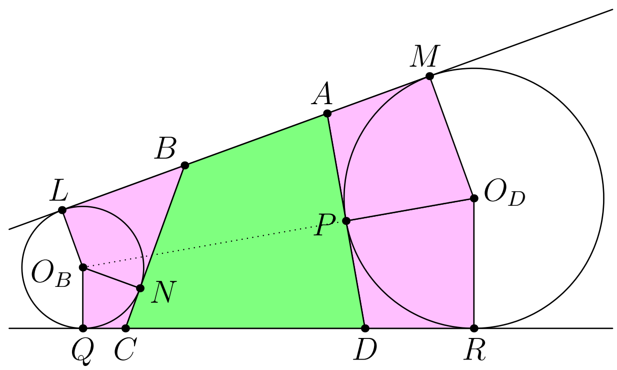
Note by symmetry that $O_BO_DRQ = O_BO_DML$.

Also note that O_BLBN can be split into triangles O_BLB and O_BNB, both with height b and bases BL and BN, which are equivalent by the Two Tangent Theorem. Thus, its area is $BN \cdot b$. Similarly, $[O_BNCQ] = CN \cdot b$, implying that $[O_BLBN] + [O_BNCQ] = (BN + CN)b = xb$. Analogously, we get $[O_DRDP] + [O_DPAM] = zd$.

Thus,

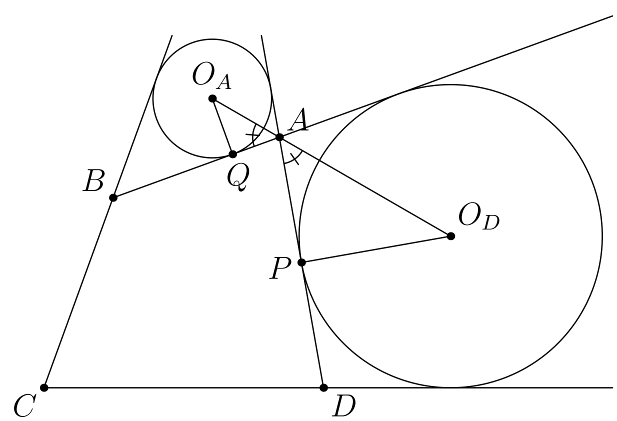
$$K = (b+d)s - xb - zd = b(s-x) + d(s-z),$$

as desired.



Claim 2: $\frac{w}{a} + \frac{y}{c} = \frac{x}{b} + \frac{z}{d}$.

Let O_D and O_A be the centers of the circles with radii d and a, respectively. Then mark points P and Q as shown below.



By the definition of the excenter, A lies on $O_D O_A$, and the angle bisector of $\angle DAB$ is perpendicular to $O_D O_A$. Thus, $\angle O_D AP = \angle O_A AQ$, implying that $\triangle O_D AP \sim \triangle O_A AQ$. Thus $\frac{AP}{d} = \frac{AQ}{a}$. You can get analogous similarities for the tangent lengths from any vertex of $ABCD$; summing them gives us our desired conclusion.

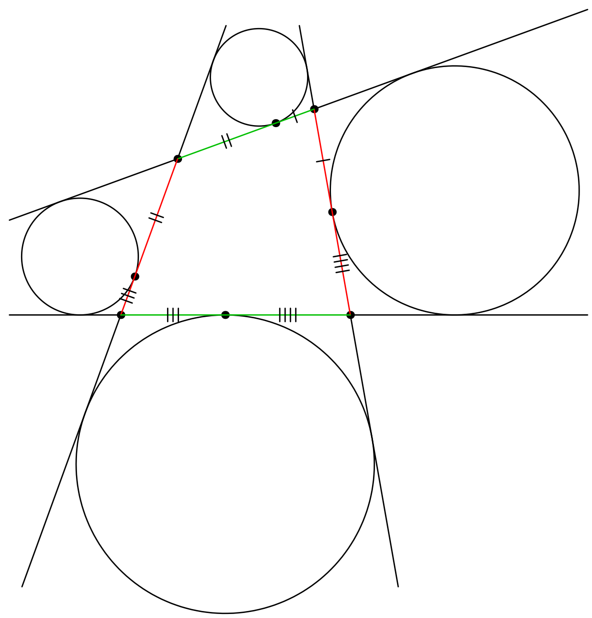


Figure 1: Marked pairs of segments are not congruent; they belong to the same pair of similar triangles. The similarities involving the green segments are summed up on one side of the equality, and the similarities involving the red segments are summed up on the other side.

Note that since $w + y = 2s$, we get $K = ay + cw = ac(\frac{w}{a} + \frac{y}{c})$. This implies that $ac = bd$, which is the key to the entire problem.

Plugging in the given radii $a = 12$, $b = 16$, $c = 28$, and $d = r$ into $ac = bd$, we obtain

$$12 \times 28 = 16 \times r \implies 336 = 16r \implies r = \boxed{21}.$$

9. **(Tiebreaker)** We choose 4 points in the space $[0, 1]^3 \subseteq \mathbb{R}^3$. The circumradius of the sphere on which those 4 points lie is r. Find the expected value of $\frac{1}{r^2+1}$.

Proposed by Henry Zheng

Answer. 0.6181

Solution. Plugging this into python (and running for a million iterations) returns an answer of about 0.6181.