Combinatorics and Computer Science Round Solutions

1. Let the set $A = \{1, 4, 9........2025\}$ where A contains all positive perfect squares up to 2025. Two elements of A are selected at random without replacement. Find the probability that their difference is a multiple of 3.

   Proposed by Emir Naduvil Valappil

   Answer. $\frac{6}{11}$

   Solution. First, note that there are 45 perfect squares, meaning there are $\binom{45}{2} = 990$ possible ways to pick perfect squares. Consider the cases where both perfect squares are multiples of 3, where one is a multiple of 3, and where neither are multiples of 3. Case 1: both multiples of 3 Then $0 - 0 \cong 0 \bmod 3$, so we can pick any combination of two perfect squares that are both multiples of 3. Since there are 15 multiples of 3 in A, we have $\binom{15}{2} = 105$ cases here. Case 2: one multiple of 3, one non-multiple of 3 Perfect squares are necessarily $\cong 1 \bmod 3$ - consider $1^2 \bmod 3 \cong 1 \bmod 3$ and $2^2 \bmod 3 \cong 4 \bmod 3 = 1$. $1 - 0 = 1 \bmod 3$ and $0 - 1 = 2 \bmod 3$, so no cases work here Case 3: two non-multiples of 3 Then $1 - 1 = 0 \bmod 3$, so we can choose any two perfect squares where neither is a multiple of 3. So we have $\binom{30}{2} = 435$ cases here. Summing all the cases, we have $105 + 0 + 435 = 540$ cases, and the probability is then $\frac{540}{990} = \boxed{\frac{6}{11}}$

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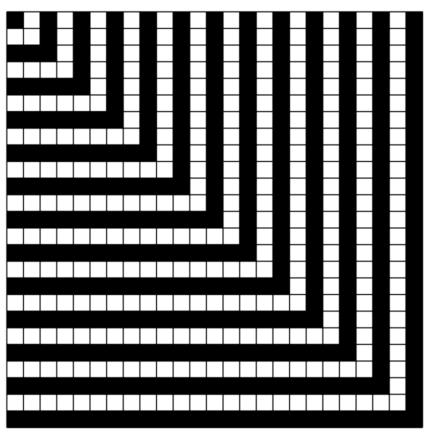
2. A 25×25 square grid is colored as follows.

   I start in the upper left corner and move according to the following rules: On black squares I can only move left or right to another square. On white squares I can only move up or down to another square. What is the total number of squares in the grid that I can access with these rules?

   Proposed by John Li

   Answer. 325

   Solution. If we look at each of the "layers" of black and white, we can see that it's possible to construct a path that passes through exactly n squares in the nth layer; whenever we arrive at a layer, we sweep either vertically or horizontally as far as we can in both directions, then transition to the next layer. Furthermore we can see that this is optimal because all other squares are inaccessible; for any white square that we haven't covered already, we would need to go up from a black square, down from a black square, or left from a white square to get to it. Similarly, the remaining black squares are inaccessible. Therefore, we can compute the total number of accessible squares as $1 + 2 + \cdots + 25 = 325$.

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3. Alice wrote 6 numbers in a ring formation. (Rotations and reflections are deemed distinct.) Each number is either 1 or 0. A jester secretly flips 2 of these numbers at random. (either $1 \to 0$ or $0 \to 1$). Alice hires Bob the investigator to help her.

   Bob has a tool that checks 3 neighboring numbers and ONLY returns how many numbers are defective. He can use this tool as much as he wants.

   What is the probability that Bob can figure out where the defects are?

   Proposed by Lohith Tummula

   Answer. $\frac{4}{5}$

   Solution. Let the positions in the ring be labeled $1, 2, 3, 4, 5, 6$ in order. Let $x_i = 1$ if the number at position i was flipped, and 0 otherwise, so $\sum_{i=1}^{6} x_i = 2$. Bob's tool checks 3 neighboring positions, returning $y_i = x_i + x_{i+1} + x_{i+2}$ for each $i = 1, \ldots, 6$ (indices mod 6). Taking differences gives $y_i - y_{i+1} = x_i - x_{i+3}$ for $i = 1, 2, 3$.

   If the two flipped numbers are adjacent or separated by distance 2, the differences $x_i - x_{i+3}$ uniquely identify the positions of the two flipped numbers. However, if the two flipped numbers are opposite (distance 3 apart, such as positions 1 and 4), then $x_i - x_{i+3} = 0$ for all i and $y_i = 1$ for all i, so Bob cannot determine which of the 3 opposite pairs $(1, 4)$, $(2, 5)$, or $(3, 6)$ was flipped.

   Out of all $\binom{6}{2} = 15$ equally likely pairs of positions that the jester could flip, exactly 3 pairs are opposite.

   Therefore, the probability that Bob can figure out where the defects are is $1 - \frac{3}{15} = 1 - \frac{1}{5} = \boxed{\frac{4}{5}}$.

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4. I'm going to write 100 numbers $a_1, \ldots, a_{100}$. For a_1 through a_6, I independently assign each of them a random integer between 0 and 6 inclusive. For $i > 6$, i write $a_i = \text{MEX}(\{a_{i-1}, \ldots, a_{i-6}\})$, where $\text{MEX}(S)$ denotes the smallest nonnegative integer which is not in the set S. The probability that $a_{100} = 6$ can be written as $\frac{m}{n}$ where $\gcd(m, n) = 1$. Find $m + n$.

   Proposed by Avnith

   Answer. 18607

   Solution. The sequence a_i for $i > 6$ cycles through each of the integers $0, \ldots, 6$. Thus the answer is the same as the probability that $a_9 = 6$. The first $i > 6$ such that $a_i = 6$ is related to the largest prefix of $[a_6, a_5, a_4, a_3, a_2, a_1]$ which is a subset of $\{0, 1, 2, 3, 4, 5\}$ without repeats. To make $a_9 = 6$, then a_6, a_5, a_4, a_3 should be a subset of $\{0, 1, 2, 3, 4, 5\}$, adding a_2 should break this, and a_1 can be anything. So the number of ways is $\binom{6}{4} \cdot 4! \cdot 5 \cdot 7 = 12600$. The total number of ways is $7^6 = 117649$, so the probability is $\frac{12600}{117649} = \frac{1800}{16807}$. So $m = 1800$ and $n = 16807$, giving $m + n = \boxed{18607}$.

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5. What is the average value of the product abc over all ordered triples of non-negative integers (a, b, c) where $a + b + c = 100$? *Proposed by Max Du*

   Answer. 16170

   Solution. To compute the sum of all abc, consider the 5-element subsets of $\{1, ..., 102\}$. Fix the second smallest element to be $a + 1$ and the second largest to be $a + b + 2$. Then, there are abc ways to choose the rest of the elements. Taking all such subsets over all choices for the 2nd and 4th elements gives all 5-element subsets, so the sum becomes just $\binom{102}{5}$. Stars and bars gives us $\binom{102}{2}$ as the number of triples, so the answer is $\frac{\binom{102}{5}}{\binom{102}{2}} = 16170$.

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6. Three airlines operate flights that fly between 2026 cities. Each flight is operated by one airline and connects two cities in both ways. The three airlines want to make sure that if any one of the airlines stops operating, each city can still be reached from any other city via some path of flights operated by the other two airlines. Compute the minimum number of total flights needed across all three airlines for this to be possible.

   Proposed by Yinuo Huang

   Answer. 3038

   Solution. Let the airlines have a, b, c flights, respectively. Note that for the 2026 cities to be connected, we need each pair of airlines two have at least $2026 - 1 = 2025$ flights together. Therefore, we have $a + b \geq 2025$, $b + c \geq 2025$, and $c + a \geq 2025$. Adding the three inequalities gives $2(a + b + c) \geq 3 \cdot 2025$, so $a + b + c \geq 3037.5$, and since a, b, c are integers, $a + b + c$ is at least $\boxed{3038}$. Note that this is possible with the following construction: first, connect cities 1 and 2 with 2 airlines. Then, we can connect any 2 new cities with 3 flights as follows: if the two new cities are X and Y, we can pick some city A already in the network, and have $A - X$ by airline 1, $A - Y$ by airline 2, and $X - Y$ by airline 3. It is easy to check that if any one of these edge is removed, X and Y are still both connected to A. By induction, we can build a connected flight graph among 2026 cities by adding pairs of new cities 1012 times on top of the 2 base cities. Since $2 + 2 \cdot 1012 = 2026$, we used $2 + 3 \cdot 1012 = 3038$ flights using this construction, as desired.

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7. For an undirected simple graph G, consider the following triangle toggling operation: Select distinct vertices u, v, w, and toggle each of $(u, v), (v, w), (w, u)$ (here, toggling means that if the edge exists, remove it, otherwise add it). What is the maximum integer E such that given any undirected simple graph G on 2025 vertices, one can apply the triangle toggling operation a finite number of times to create a graph with at least E edges?

   Proposed by James Yang **Answer.** 2048288

   Solution. Lower bound: If one greedily applies the operation to maximize the number of edges, it is clear that upon halting, the resulting graph does not have pairs $(u, v), (u, w)$ which both don't exist. In other words, the set of non-existent edges form a matching. This means the graph at this point will have at least $\binom{2025}{2} - \lfloor \frac{2025}{2} \rfloor = 2048288$ edges.

   Upper bound: Note that the operation does not change the parity of degrees of vertices. Therefore, if we choose the starting G such that 2024 vertices have odd degree, then the maximum possible sum of degrees after applying some triangle toggling operations is $2024 \times 2023 + 1 \times 2024 = 4096576$. Using handshake lemma, this means there can be at most $\frac{4096576}{2} = 2048288$ edges in this case.

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8. Consider $S = \{1, 2, 3, \cdots, 2027^2\}$. Find the number of 2026-sized subsets of S, such that the product of the elements in the subset is $1 \mod 2027$.

   Proposed by Henry Zheng, Solution by Henry Zheng and Wilson Pan

   Answer. $\frac{1}{2026}\left[\binom{2026\times 2027}{2026} - \binom{1013\times 2027}{1013} + 1012\binom{4054}{2} - 1012(2027)\right]$

Solution. Let $p = 2027$. Our starting set is $S = \{1, 2, 3, \ldots, p^2\}$. We want to choose subsets of size $p-1$ such that the product of the elements is $\equiv 1 \pmod p$. We must discard any element that is a multiple of p. If a subset contains even one multiple of p, the entire product becomes $0 \pmod p$. There are exactly p multiples of p in our set $(p, 2p, \ldots, p^2)$. Removing these leaves a valid pool of $p^2 - p = p(p-1)$ elements. Notice each non-zero residue $(1, 2, \ldots, p-1)$ appears exactly p times.
We can define our generating function as

$$P(x, y) = \prod_{k=0}^{p-2} (1 + yx^k)^p.$$

The coefficient of $y^n x^s$ in $P(x, y)$ is equivalent to the number of subsets of size n and sum s. A way to interpret our generating function is for each k, $(1 + y^k)$ is the choice between including an element $\equiv k \pmod p$ or not. There are p total elements congruent to k so we raise to p power.
Let $f(x)$ be the polynomial representing the choices of exactly $p-1$ elements:

$$f(x) = [y^{p-1}]P(x, y). \qquad \text{(Terms with } y^{p-1} \text{ in } P(x, y))$$

We want the sum of the coefficients of x^N in $f(x)$ where $N \equiv 0 \pmod{p-1}$.
We can use roots of unity filter of $\omega = e^{2\pi i/(p-1)}$.
The sum S is:

$$S = \frac{1}{p-1} \sum_{j=0}^{p-2} f(\omega^j)$$

We need to evaluate $P(\omega^j, y)$ for each j.

$$P(\omega^j, y) = \prod_{k=0}^{p-2} (1 + y\omega^{jk})^p.$$

Let $d = \gcd(j, p-1)$ and let $m = (p-1)/d$. As k cycles from 0 to $p-2$, the term ω^{jk} cycles through the m-th roots of unity, hitting each one exactly d times. Therefore:

$$P(\omega^j, y) = \left(\prod_{r=0}^{m-1} \left(1 + ye^{2\pi i r/m}\right) \right)^{dp}$$

Using $\prod_{r=0}^{m-1}(1 + ye^{2\pi i r/m}) = 1 - (-1)^m y^m$ (derived from writing $x^n - 1 = 0$ in terms of it's roots),

$$P(\omega^j, y) = (1 - (-1)^m y^m)^{dp}.$$

$f(\omega^j)$ is the coefficient of y^{p-1} in this expansion. Since $p - 1 = md$, we need the coefficient of y^{md}, which is:

$$c_d = \binom{dp}{d} (-(-1)^m)^d = \binom{dp}{d} (-1)^{d(m+1)}.$$

Notice that c_d depends entirely on $d = \gcd(j, p-1)$. The number of times a specific d occurs as j ranges from 0 to $p-2$ is given by Euler's totient function, $\phi(m)$.
Plug in $p = 2027$, which means $p - 1 = 2026$. The divisors of 2026 are $d \in \{1, 2, 1013, 2026\}$. The term for each divisor would be

(a) For $d = 2026$: $m = 1$. $\phi(1) = 1$.

$$c_{2026} = \binom{2026 \times 2027}{2026} (-1)^{2026(2)} = \binom{2026 \times 2027}{2026}$$

(b) For $d = 1013$: $m = 2$. $\phi(2) = 1$.

$$c_{1013} = \binom{1013 \times 2027}{1013} (-1)^{1013(3)} = -\binom{1013 \times 2027}{1013}$$

(c) For $d = 2$: $m = 1013$. $\phi(1013) = 1012$ (since 1013 is prime).

$$c_2 = \binom{2 \times 2027}{2}(-1)^{2(1014)} = \binom{4054}{2}$$

(d) For $d = 1$: $m = 2026$. $\phi(2026) = 1012$.

$$c_1 = \binom{1 \times 2027}{1}(-1)^{1(2027)} = -2027$$

So our final sum is

$$S = \frac{1}{2026}\left[\binom{2026 \times 2027}{2026} - \binom{1013 \times 2027}{1013} + 1012\binom{4054}{2} - 1012(2027)\right].$$