CMIMC

Combinatorics Round

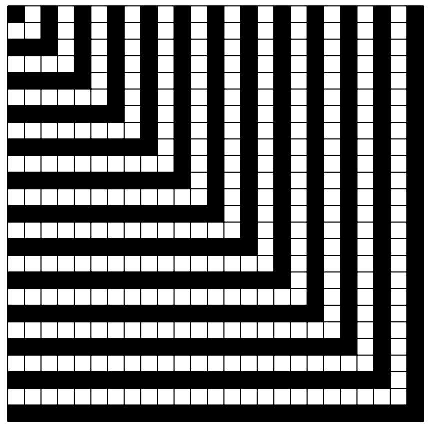
Instructions

1. Do not look at the test before the proctor starts the round.
2. This test consists of 8 short-answer problems to be solved in 50 minutes. The final estimation question will be used to break ties.
3. No computational aids other than pencil/pen are permitted. You may NOT communicate with your teammates.
4. Write your name, team name, and competitor ID on your answer sheet.
5. Write your answers in the corresponding lines on your answer sheet.
6. Answers must be reasonably simplified.
7. If you believe that the test contains an error, submit your protest to the 2026 CMIMC Discord.
8. Suspicion of cheating may render disqualification of scores and results.

Combinatorics

1. Let the set $A = \{1, 4, 9........2025\}$ where A contains all positive perfect squares up to 2025. Two elements of A are selected at random without replacement. Find the probability that their difference is a multiple of 3.

2. A 25×25 square grid is colored as follows.



   I start in the upper left corner and move according to the following rules: On black squares I can only move left or right to another square. On white squares I can only move up or down to another square. What is the total number of squares in the grid that I can access with these rules?

3. Alice wrote 6 numbers in a ring formation. (Rotations and reflections are deemed distinct.) Each number is either 1 or 0. A jester secretly flips 2 of these numbers at random. (either $1 \to 0$ or $0 \to 1$). Alice hires Bob the investigator to help her.

   Bob has a tool that checks 3 neighboring numbers and ONLY returns how many numbers are flipped by the jester. He can use this tool as much as he wants.

   What is the probability that Bob can figure out where the defects are?

4. I'm going to write 100 numbers $a_1, \ldots, a_{100}$. For a_1 through a_6, I independently assign each of them a random integer between 0 and 6 inclusive. For $i > 6$, i write $a_i = \mathrm{MEX}(\{a_{i-1}, \ldots, a_{i-6}\})$, where $\mathrm{MEX}(S)$ denotes the smallest nonnegative integer which is not in the set S. The probability that $a_{100} = 6$ can be written as $\frac{m}{n}$ where $\gcd(m, n) = 1$. Find $m + n$.

5. What is the average value of the product abc over all ordered triples of non-negative integers (a, b, c) where $a + b + c = 100$?

6. Three airlines operate flights that fly between 2026 cities. Each flight is operated by one airline and connects two cities in both ways. The three airlines want to make sure that if any one of the airlines stops operating, each city can still be reached from any other city via some path of flights operated by the other two airlines. Compute the minimum number of total flights needed across all three airlines for this to be possible.

7. For an undirected simple graph G, consider the following triangle toggling operation: Select distinct vertices u, v, w, and toggle each of $(u, v), (v, w), (w, u)$ (here, toggling means that if the edge exists, remove it, otherwise add it). What is the maximum integer E such that given any undirected simple graph G on 2025 vertices, one can apply the triangle toggling operation a finite number of times to create a graph with at least E edges?

8. Consider $S = \{1, 2, 3, \cdots, 2027^2\}$. Find the number of 2026-sized subsets of S, such that the product of the elements in the subset is $1 \mod 2027$.

9. **Tiebreaker:** Suppose we are doing a random walk over a 2-D lattice, where we start at the origin. What is the probability that somwhere in the first 1000 steps, we have been through the point $(2, 2)$? Give your answer in the form $0.abcdef$.