Algebra and Number Theory Round Solutions

1. You are handing out some positive integer number of candies to Alice, Bob, and Carol, and they each receive $\frac{1}{4}$, $\frac{1}{6}$, and $\frac{1}{8}$ of the total number of candies, respectively. Afterwards, you realize the leftover candies can be split evenly between Alice, Bob, Carol, and yourself. What's the lowest number of candies you could've started with?

   Proposed by John Li

   Answer. 96

   Solution. Let the starting number of candies be N. Then, the number of leftover candies is $N - \frac{N}{4} - \frac{N}{6} - \frac{N}{8} = \frac{11N}{24}$. We want this number to be divisible by 4, which implies that $\frac{N}{24}$ must also be divisible by 4. The case resulting in the least N occurs when $\frac{N}{24} = 4$, so $N = 96$.

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2. The average temperature in Pittsburgh, PA over the past 12 days was $23°F$. If no temperature occurred more than twice, every temperature was an integer, and the minimum temperature was $15°F$, find the maximum possible temperature in any of these 12 days.

   Proposed by Emir Naduvil Valappil

   Answer. 86

   Solution. The sum of the temperatures is 276. Maximize the number of days where the temperature is $15, 16, 17, 18, 19°F$. This contributes a total of $2 \cdot (15 + 16 + 17 + 18 + 19) = 170°F$ degrees. Then add in a $20°F$day, and then there are $276 - 170 - 20 = \boxed{86°F}$ to make the average $23°F$.

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3. Let k be the sum of the first $2^{89} - 1$ positive integers. It is a fact that k has 178 factors. If n is the sum of the first $2^{89} - 1$ positive cubes, how many factors does n have?

   Proposed by Eric Oh

   Answer. 531

   Solution. Note that $k = 2^{88} * (2^{89} - 1)$ since that has 178 factors, $2^{89} - 1$ has to be prime. Also, $n = k^2$ so $n = 2^{176} * p^2$ for $p = 2^{89} - 1$, so that has $177 * 3 = 531$ factors.

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4. Consider the arithmetic sequence $a_1, a_2, ..., a_{43}$ with common difference of $d > 0$ and $a_{22} = 1$. Given that

   $$\frac{1}{a_1 \cdot a_2 \cdot a_3} + \frac{1}{a_2 \cdot a_3 \cdot a_4} + ... + \frac{1}{a_{41} \cdot a_{42} \cdot a_{43}} = 41,$$

   compute d.

   Proposed by Minerva You

Answer. $\frac{29}{420}$

Solution. Note

$$\frac{1}{a_k a_{k+1} a_{k+2}} = \frac{1}{2d}\left(\frac{1}{a_k a_{k+1}} - \frac{1}{a_{k+1} a_{k+2}}\right).$$

Thus we have

$$\begin{aligned}\frac{1}{a_1 \cdot a_2 \cdot a_3} + \frac{1}{a_2 \cdot a_3 \cdot a_4} + ... + \frac{1}{a_{41} \cdot a_{42} \cdot a_{43}} &= \frac{1}{2d}\left(\frac{1}{a_1 a_2} - \frac{1}{a_2 a_3} + \frac{1}{a_2 a_3} - \frac{1}{a_3 a_4} + \cdots + \frac{1}{a_{41} a_{42}} - \frac{1}{a_{42} a_{43}}\right) \\ &= \frac{1}{2d}\left(\frac{1}{a_1 a_2} - \frac{1}{a_{42} a_{43}}\right) \\ &= \frac{1}{2d}\left(\frac{1}{(1-21d)(1-20d)} - \frac{1}{(1+21d)(1+20d)}\right) = 41 \\ \implies d &= \frac{29}{420}\end{aligned}$$

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5. Find the largest prime p such that $\sqrt{1024 + 3p}$ is an integer.

   Proposed by Avnith Vijayram

   Answer. 67

   Solution. Let $32 + k = \sqrt{1024 + 3p}$. Squaring both sides, $1024 + 64k + k^2 = 1024 + 3p$ so $k(k + 64) = 3p$. Since $k > 0$ and p is prime and we can infer $p > 3$, this implies that k is either 1 or 3. Checking both values, we find that $k = 3$ works with $p = 67$.

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6. How many multiples of 3 are in the 2026th row of Pascal's Triangle, i.e. the row with 2027 entries? (We number the rows starting from 0.)

   Proposed by Maxwell Du

   Answer. 1991

   Solution. Consider the triangle mod 3. Notice that the 3^nth row contains 1s on the sides, and the $2 \cdot 3^n$th row contains 1s on the sides again and a 2 in the center. What ends up happening is that under the 1s there is a copy of the first 3^n rows, and under the 2 there is also a copy of the rows but doubled. Thus, every 1 in the row number's base-3 expansion contributes a factor of 2 and every 2 contributes a factor of 3 to the number of nonzero entries.

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7. Consider the solutions to

$$3x^2 + 4xy + y^2 + 2x + 4y \equiv 1 \pmod{8}3$$

   where x, y are nonnegative integers less than 83. For how many values of x does there exist a solution for y?

Answer. 42

Solution. For some K we will decide later, we can use the quadratic formula on

$$3x^2 + 4xy + y^2 + 2x + 4y + K = 0$$

to get that $x = \frac{-4y-2\pm\sqrt{16y^2+16y+4-12y^2-48y-12K}}{6}$ The discriminant is $4y^2 - 32y + 4 - 12K$. We want this to be a perfect square, so we set $K = -5$ to get that $x = \frac{-4y-2\pm(2y-8)}{6}$. So this motivates the factorization $3x^2 + 4xy + y^2 + 2x + 4y - 5 = (3x + y + 5)(x + y - 1)$

Hence, we want to solve

$$(x + y - 1)(3x + y + 5) \equiv -4 \pmod 83$$

If we let $z = x + y - 1$, then we want to find solutions to

$$z(z + 2x + 6) \equiv -4 \pmod 83$$

$$z^2 + 2xz + 6z \equiv -4 \pmod 83$$

$$x \equiv -\left(\frac{z}{2} + 3 + \frac{2}{z}\right) \pmod 83$$

Clearly z cannot be $0 \pmod 83$. Moreover, for any other z, there is a solution to x, y given by $x \equiv -\left(\frac{z}{2} + 3 + \frac{2}{z}\right) \pmod 83$ and $y \equiv z - x + 1 \pmod 83$. Moreover, note that if we replace z with z^{-1}, we get the same value for x. The only elements that are their own inverse in $\pmod 83$ are ± 1, so the number of unique. One can then show that if there exists z, z' such that given $f(z) = \frac{z}{2} + 3 + \frac{2}{z}$ we have $f(z) = f(z')$ then either $z \equiv z' \pmod 83$ or $z^{-1} \equiv z' \pmod 83$. Hence, the number of unique x is just $1 + 1 + \frac{80}{2} = 42$.

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8. Compute $\tan^4(20°) + \tan^4(40°) + \tan^4(60°) + \tan^4(80°)$.

   Answer. 1044

   Solution. Notice this equals $\tan^4(40°) + \tan^4(80°) + \tan^4(120°) + \tan^4(160°)$. We wish to find a polynomial with roots $\tan(40°), \tan(80°), \ldots, \tan(360°)$. Suppose $x = \tan\alpha$. Consider $(1 + xi)^9$. This has argument 9α, which we want to equal 0. We will then set the imaginary part of the number to 0, getting us $9x - \binom{9}{3}x^3 + \binom{9}{5}x^5 - \binom{9}{7}x^7 + x^9 = 0$. Divide by x and think of this as a poly in x^2. This gets us $\binom{9}{2}^2 - \binom{9}{4} = 1044$.

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9. **(Tiebreaker)** For a positive integer n, define $f(n)$ to be the sum of the positive factors of n. Estimate $f(1) + \cdots + f(10,000)$.

   Proposed by Henry Zheng

   Answer. 82256014

   Solution. A Python script returns the exact answer.