

Team Round Solutions

1. I define a “good day” as a day when both the day and the month evenly divide the concatenation of the two. For example, today (March 15) is a good day since 3 and 15 both divide 315. However, March 9 is not a good day since 9 does not divide 39. How many good days are in March, April, and May combined?

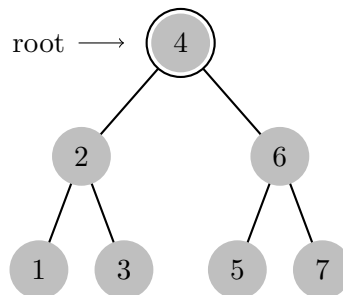
Proposed by Lohith Tummala

Answer. 13

Solution. Let d be the day and m be the month. To test if a day is a good day, check if d is a multiple of m , and that d is a factor of $100m$ (or $10m$ in the case of a single digit day). If $m = d$, then we have a good day. We can also get a good day by multiplying d by factors of 100 as well. Counting, we get 13:

$$3/3, 3/6, 3/12, 3/15, 3/30, 4/4, 4/8, 4/16, 4/20, 5/5, 5/10, 5/20, 5/25$$

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2. We are searching for the number 7 in the following binary tree:



We use the following algorithm (which terminates with probability 1):

1. Write down the number currently at the root node
2. If we wrote down 7, terminate
3. Else, pick a random edge, and swap the two numbers at the endpoints of that edge
4. Go back to step 1

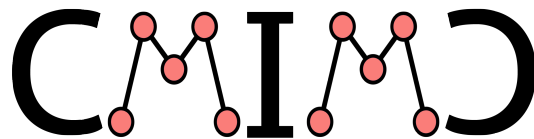
Let $p(a)$ be the probability that we ever write down the number a after running the algorithm once. Find

$$p(1) + p(2) + p(3) + p(5) + p(6).$$

Proposed by Justin Hsieh

Answer. $\frac{11}{4}$

Solution. First, we have $p(1) = p(3) = p(5) = \frac{1}{2}$. By symmetry, there is a $\frac{1}{2}$ chance that 1 gets to the root before 7 does. The same holds for 3 and 5.



Next, we compute $p(2)$. Consider the first time either 2 or 7 is moved. There is a $\frac{1}{4}$ chance that this move brings 2 to the root, and a $\frac{3}{4}$ chance that this move brings 2 and 7 to the same level, at which point we can apply symmetry. This gives us $p(2) = \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{2} = \frac{5}{8}$.

Finally, we compute $p(6)$. For this case, we can consider the first time 6 or 7 is moved. There is a $\frac{1}{3}$ chance that 6 is moved to the root, a $\frac{1}{3}$ chance that 6 is moved to the same level as 7, and a $\frac{1}{3}$ chance that 6 and 7 are swapped. We get

$$p(6) = \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3}(1 - p(6)),$$

and we can solve this equation to get $p(6) = \frac{5}{8}$.

In total,

$$p(1) + p(2) + p(3) + p(5) + p(6) = \frac{1}{2} + \frac{5}{8} + \frac{1}{2} + \frac{1}{2} + \frac{5}{8} = \boxed{\frac{11}{4}}.$$

3. Let $f(x) = x^4 - 4x^2 + 2$. Find the smallest natural $n \in \mathbb{N}$ such that there exists $k, c \in \mathbb{N}$ with

$$\left| f^k \left(\frac{n^2 + 1}{n} \right) - c^{144} \right| < \frac{1}{100}.$$

Proposed by James Yang

Answer. 512

Solution. Note that in general, $f(n + 1/n) = n^4 + 1/n^4$, so in general, we have the formula

$$f^k(n + 1/n) = n^{4^k} + 1/n^{4^k}$$

so we want $n^{4^k} = c^{144}$. Since $144 = 4^2 \cdot 9$, then we just need n to be a 9-th power. It's clear then that the answer is $2^9 = \boxed{512}$.

4. A non-self intersecting hexagon $RANDOM$ is formed by assigning the labels R, A, N, D, O, M in some order to the points

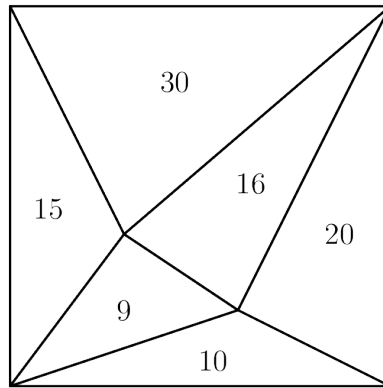
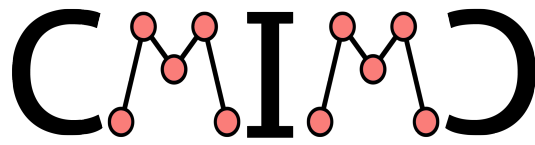
$$(0, 0), (10, 0), (10, 10), (0, 10), (3, 4), (6, 2).$$

Let $a_{\max}$ be the greatest possible area of $RANDOM$ and $a_{\min}$ the least possible area of $RANDOM$. Find $a_{\max} - a_{\min}$.

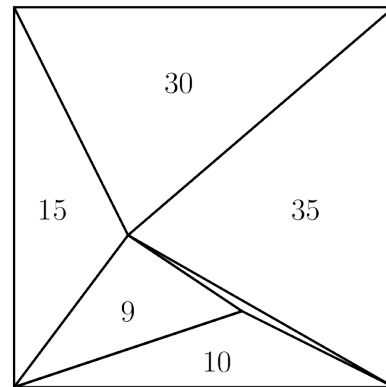
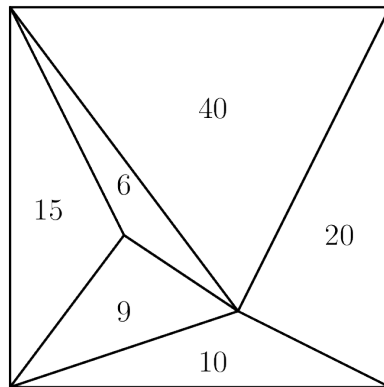
Proposed by Justin Hsieh

Answer. 44

Solution. We split up the square $(0, 0), (10, 0), (10, 10), (0, 10)$ into triangles, with $(3, 4), (6, 2)$ as additional vertices:



Each triangle is labelled with its area. There are two more possible configurations that can be obtained by flipping an edge:



(The unlabelled triangle in the right diagram has area 1.) There are more ways to split the square into triangles, but these involve creating an edge between opposite corners of the square (between $(0,0)$ and $(10,10)$, or between $(10,0)$ and $(0,10)$). Such an edge cannot be part of a non-self intersecting hexagon because it separates the other pair of opposite corners.

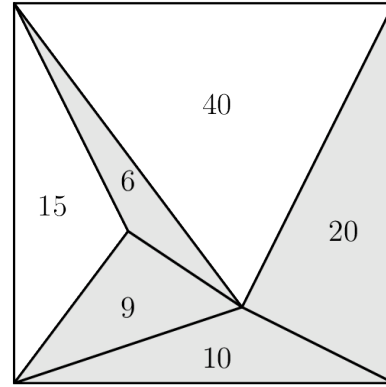
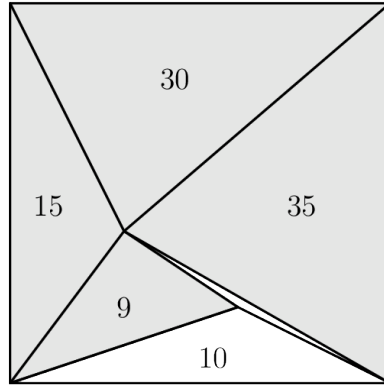
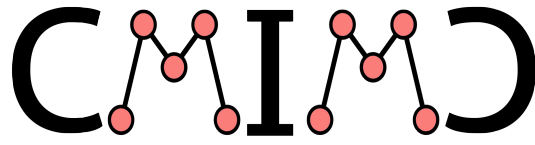
Every possible non-intersecting hexagon can be formed by removing two triangles from one of the squares above. To find the maximum area, we want to remove two triangles of small area. The two smallest triangles are in the third configuration, with areas 1 and 9. This is not a valid hexagon, but the next smallest areas 1 and 10 give the hexagon

$$(0,0), (6,2), (3,4), (10,0), (10,10), (0,10).$$

To find the minimum area, we want to remove two triangles of large area. The largest triangles that give a valid hexagon are the triangles of area 40 and 15 in the second configuration. This gives the hexagon

$$(0,0), (10,0), (10,10), (6,2), (0,10), (3,4).$$

The maximum possible area is $a_{\max} = 89$, the minimum possible area is $a_{\min} = 44$, and the answer is $89 - 45 = \boxed{44}$.



5. Suppose we have a uniformly random function from $\{1, 2, 3, \dots, 25\}$ to itself. Find the expected value of

$$\sum_{x=1}^{25} (f(f(x)) - x)^2.$$

Proposed by Ishin Shah

Answer. 2496

Solution. We will define X_k to be the random variable of $X = (f(f(k)) - k)^2$. By linearity of expectation, we need to find $\sum_{k=1}^n E[X_k]$. Thus, we will find each individual X_k .

For each k , we will case upon if $f(k) = k$.

$$E[X_k] = E[X_k | f(k) = k]P[f(k) = k] + E[X_k | f(k) \neq k]P[f(k) \neq k]$$

If $f(k) = k$, then $X_k = 0$. Otherwise, we also have $P[f(k) \neq k] = \frac{n-1}{n}$.

Now, if $f(k) \neq k$, for the function evaluated at $f(k)$ has a uniformly random choice of functions. Thus, $E[X_k | f(k) \neq k] = \frac{1}{n} \sum_{i=1}^n (k-i)^2$

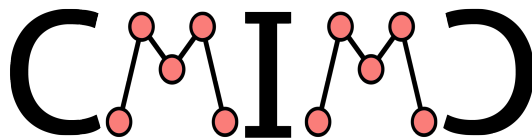
Thus, $E[X_k] = \frac{n-1}{n^2} \sum_{i=1}^n (k-i)^2$.

Our final solution is

$$\frac{n-1}{n^2} \sum_{k=1}^n \sum_{i=1}^n (k-i)^2 = \frac{n-1}{n^2} \sum_{k=1}^n \sum_{i=1}^n k^2 - 2ki + i^2.$$

We can split this up into

$$\begin{aligned} & \frac{n-1}{n^2} \left(\sum_{k=1}^n \sum_{i=1}^n 2k^2 - \sum_{k=1}^n \sum_{i=1}^n 2ki \right) = \\ & \frac{2(n-1)}{n} \sum_{k=1}^n k^2 - \frac{2(n-1)}{n^2} \left(\sum_{k=1}^n k \right)^2 = \\ & \frac{2(n-1)}{n} \frac{n(n+1)(2n+1)}{6} - \frac{2(n-1)}{n^2} \left(\frac{n(n+1)}{2} \right)^2 = \end{aligned}$$



$$\frac{(n-1)(n+1)(2n+1)}{3} - \frac{(n-1)(n+1)^2}{2}$$

Plugging in $n = 25$ gets the answer of $\boxed{2496}$.

6. Suppose we have a regular 24-gon labeled $A_1 \cdots A_{24}$. We will draw 2 smaller regular 24-gons within $A_1 \cdots A_{24}$. For the sake of this problem, make $A_i = A_{i+24}$.

With our first configuration, we create 3 stars by creating lines $\overline{A_i A_{i+9}}$. A 24-gon will be created in the center, which we denote as our first 24-gon.

With our second configuration, we create a star by creating lines $\overline{A_i A_{i+11}}$. A 24-gon will be created in the center, which we denote as our second 24-gon.

Find the ratio of the areas of the first 24-gon to the second 24-gon.

Proposed by Ishin Shah

Answer. $3 + \sqrt{2} + \sqrt{3} + \sqrt{6}$ or $3 + \sqrt{3} + 2\sqrt{2 + \sqrt{3}}$

Solution. Let O be the center of $A_1 \cdots A_{24}$. Also, let $r = OA_i$ for some i . The center of the new 24-gons will be O by symmetry of the regular 24-gons. Now, we will find the ratio of the sizes of the polygons. We will do this by finding the distance from O to the lines that create the lines.

Now, analyzing the triangles $OA_i A_{i+9}$ gives a $22.5^\circ - 22.5^\circ - 135^\circ$ triangle. This makes the distance from O to the sides of our first 24-gon equal to $r \sin(22.5^\circ)$.

Also, triangles $OA_i A_{i+11}$ gives a $7.5^\circ - 7.5^\circ - 165^\circ$ triangle. This makes the distance from O to the sides of our first 24-gon equal to $r \sin(7.5^\circ)$.

This makes our side length ratio equal to $\frac{\sin(22.5^\circ)}{\sin(7.5^\circ)}$.

Now, since $\sin(3x) = 3 \sin(x) - 4 \sin^3(x)$, we get $\frac{\sin(3x)}{\sin(x)} = 3 - 4 \sin^2(x) = 1 + 2 \cos(2x)$. This makes our side length ratio equal to $1 + 2 \cos(15^\circ)$.

Now, since $\cos(15^\circ) = \frac{\sqrt{6} + \sqrt{2}}{4}$, the ratio is $\frac{2 + \sqrt{6} + \sqrt{2}}{2}$.

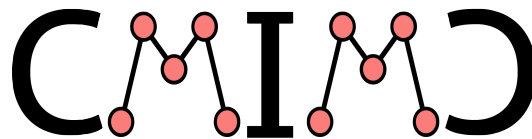
Finally, since we need to square this, we get $(\frac{2 + \sqrt{6} + \sqrt{2}}{2})^2 = \frac{1}{2}(\sqrt{3} + \sqrt{2} + 1)^2 = \boxed{3 + \sqrt{2} + \sqrt{3} + \sqrt{6}}$.

Alternatively, we could get $(1 + 2 \cos(15^\circ))^2 = 4 \cos^2(15^\circ) + 4 \cos(15^\circ) + 1 =$

$$2 \cos(30^\circ) + 4 \cos(15^\circ) + 3 = 3 + \sqrt{3} + 4 \cos(15^\circ).$$

This simplifies to $\boxed{3 + \sqrt{2} + \sqrt{3} + \sqrt{6}}$. Also, with half angle formula, we get $\cos(15^\circ) = \frac{1}{2} \sqrt{2 + \sqrt{3}}$

giving the other form of $\boxed{3 + \sqrt{3} + \sqrt{2 + \sqrt{3}}}$.



7. The binomial coefficient $\binom{n}{k}$ can be defined as the coefficient of x^k in the expansion of $(1+x)^n$. Similarly, define the trinomial coefficient $\binom{n}{k}_3$ as the coefficient of x^k in the expansion of $(1+x+x^2)^n$.

Determine the number of integers k with $0 \leq k \leq 4048$ such that $\binom{2024}{k}_3 \equiv 1 \pmod{3}$.

Proposed by Alan Abraham

Answer. 648

Solution. We see that in $\mathbb{F}_3[x]$, $(1+x+x^2) = \frac{x^3-1}{x-1} = \frac{(x-1)^3}{x-1} = (x-1)^2$. So $\binom{2024}{k}_3$ is equivalent mod 3 to the coefficient of x^k in $(x-1)^{4048}$. So our problem is equivalent to finding all k such that $\binom{4048}{k} \equiv (-1)^k \pmod{3}$. To this end we use Lucas's theorem. In base 3, $4048 = 12112221_3$. Let k have ternary representation $a_3b_3a_2a_1b_2b_1b_0a_0$ where for all i , $a_i \in \{0,1\}$ and $b_i \in \{0,1,2\}$ (if $a_i = 2$, then Lucas's theorem would tell us $3 \mid \binom{4048}{k}$). Then

$$\binom{4048}{k} \equiv \prod_{i=0}^3 \binom{1}{a_i} \binom{2}{b_i} \pmod{3} \equiv \prod_{i=0}^3 (-1)^{b_i} \pmod{3}$$

$$(-1)^k = (-1)^{\sum_{i=0}^3 a_i + b_i} = \prod_{i=0}^3 (-1)^{a_i} (-1)^{b_i}$$

So it is a necessary and sufficient condition that $\prod_{i=0}^3 (-1)^{a_i} = 1$. So for any of the $2^3 3^4$ choices of $a_1, a_2, a_3, b_0, b_1, b_2, b_3$, there will be exactly one choice for a_0 that makes this equation true. Hence, our total count is $2^3 3^4 = \boxed{648}$

8. Let U be the set of all possible complex numbers m so that the 4 roots of $(x^2 + 2x + 5)(x^2 - 2mx + 25) = 0$ are concyclic in the complex plane. One can show that when the points of U are plotted on the complex plane, it is visualized as the finite union of some curves. Find the sum of the length of these curves (i.e. the perimeter of U).

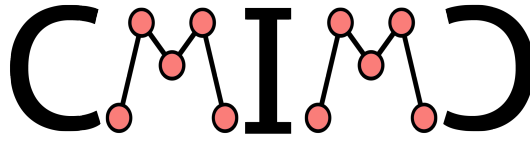
Proposed by Alan Abraham

Answer. $10 + \frac{40\pi}{3}$

Solution. Let the roots of $x^2 + 2x + 5$ be $r_1 = -1 - 2i$ and $r_2 = -1 + 2i$. Let the roots of $x^2 - 2mx + 25$ be s_1, s_2 . For r_1, r_2, s_1, s_2 to be concyclic, it is necessarily the case that the circumcenter lies on the intersection of the perpendicular bisectors of r_1, r_2 and s_1, s_2 . If this point is equally distant to all 4 points, then they are concyclic. The perpendicular bisector of r_1, r_2 is just the real axis. For the first case: if s_1 and s_2 are also conjugate pairs, then the perpendicular bisector of s_1, s_2 is also the real line. In this case r_1, r_2, s_1, s_2 form an isosceles trapezoid which is always concyclic. This happens when m is real and $4m^2 - 100 < 0$ i.e. $m \in (-5, 5)$.

For the second case: Otherwise, the perpendicular bisectors must have a unique intersection. Suppose that is $a \in \mathbb{R}$. The distance squared from a to r_1 or r_2 will be $(a+1)^2 + 4$. The distance from a to s_i is $(a-s_i)(a-\bar{s}_i)$ and must also be equal to $(a+1)^2 + 4$. This means $2a+5 = -a(s_i + \bar{s}_i) + s_i \bar{s}_i$. Note that by viete's $s_1 s_2 = 25$. Let $s = s_1$. This means we have the system of equations

$$\begin{cases} 2a+5 = -a(s+\bar{s}) + s\bar{s} \\ 2a+5 = -a\left(\frac{25}{s} + \frac{25}{\bar{s}}\right) + \frac{625}{s\bar{s}} \end{cases}$$



$$\begin{cases} 2a + 5 = -a(s + \bar{s}) + s\bar{s} \\ (2a + 5)s\bar{s} = -25a(s + \bar{s}) + 625 \end{cases}$$

Let $S = s + \bar{s}$ and $P = s\bar{s}$. Then we have

$$\begin{cases} 2a + 5 = -aS + P \\ 625 = 25aS + (2a + 5)P \end{cases}$$

This is a linear system of equations. When $a = 0$ there are no solutions. When $a = -15$, both equations are multiples of each other. Otherwise, we get that $P = 25$ and $S = \frac{20-2a}{a}$. Note that this means $\bar{s} = \frac{25}{s} = s_2$. That is s_2, s_1 are conjugate pairs which is the first case. When $a = -15$ we have that as long as $15S + P = -25$ there is a solution for s . We also must have P positive, which means $S < -\frac{5}{3}$. This means s and $\bar{s}$ are roots of the quadratic $x^2 - Sx - 15S - 25$. So S is necessarily real and $S^2 + 60S + 100 < 0$. This means that s can lie anywhere on the circle parametrized by $(t, \pm\sqrt{-t^2 - 30t - 25})$ where t ranges from all real values satisfying $-t^2 - 30t - 25 \geq 0$, which is also the circle $(x + 15)^2 + y^2 = 200$ (if we think of $\mathbb{C}$ as $\mathbb{R}^2$). By symmetry, s_2 will also lie on the same circle. Since $s_2 = \frac{25}{s_1}$, we can construct s_2 by drawing a line passing through the origin and s_1 , and then taking the conjugate of the other intersection point with $(x + 15)^2 + y^2 = 200$.

Switching our perspective to vectors in $\mathbb{R}^2$, let $u = (u_x, u_y)$ be a unit vector pointing in the direction of s_1 . Consider the solutions to t in the quadratic $(tu - p)^T(tu - p) = 200$ where $p = (-15, 0)$. This means $t^2 - 2tu^T p + p^T p - 200 = 0$. If t_1, t_2 are the roots, then $s_1 = t_1 u$ and $s_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} t_2 u$. So

$$\begin{aligned} \frac{s_1 + s_2}{2} &= \frac{1}{2} \begin{pmatrix} t_1 + t_2 & 0 \\ 0 & t_1 - t_2 \end{pmatrix} u \\ m &= \begin{pmatrix} u^T p & 0 \\ 0 & \pm\sqrt{(u^T p)^2 - p^T p + 200} \end{pmatrix} u \\ m &= \begin{pmatrix} u^T p & 0 \\ 0 & \pm\sqrt{(u^T p)^2 - 25} \end{pmatrix} u \\ -\frac{m}{15} &= (u_x^2, \pm u_y \sqrt{u_x^2 - \frac{1}{9}}) \end{aligned}$$

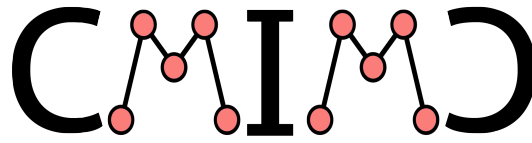
Note that

$$\begin{aligned} &\left(u_x^2 - \frac{5}{9}\right)^2 + u_y^2 \left(u_x^2 - \frac{1}{9}\right) \\ &= u_x^4 - \frac{10}{9}u_x^2 + \frac{25}{81} + u_y^2 u_x^2 - \frac{1}{9}u_y^2 \\ &= u_x^4 - \frac{10}{9}u_x^2 + \frac{25}{81} + u_x^2 - u_x^4 - \frac{1}{9} + \frac{1}{9}u_x^2 \\ &= \frac{16}{81} \end{aligned}$$

So in this case, m must be on a circle with radius $15 \cdot \frac{4}{9} = \frac{20}{3}$ centered around $-15 \cdot \frac{5}{9} = -\frac{25}{3}$. Using a continuity argument (alternatively note that u_x^2 can be any value between $\frac{1}{9}$ and 1), we see that all such values of m are attainable.

Hence, U consists of a line segment from -5 to 5 and a circle of radius $\frac{20}{3}$, so the total length is

$10 + \frac{40\pi}{3}$



9. Given a triangle, $AB = 78, BC = 50, AC = 112$, construct squares $ABXY, BCPQ, ACMN$ outside the triangle. Let L_1, L_2, L_3 be the midpoints of $\overline{MP}, \overline{QX}, \overline{NY}$, respectively. Find the area of $L_1L_2L_3$.

Proposed by Henry Zheng

Answer. 8222

Solution. Let's first try to find the length of circumradius. We can find the area by heron's formula as:

$$[ABC] = \sqrt{s(s-a)(s-b)(s-c)}$$

We get that the semiperimeter is just $(78 + 50 + 112)/2 = 120$ Therefore, the area comes out to be:

$$[ABC] = \sqrt{(120)(8)(70)(42)} = 10\sqrt{(12)(8)(7)(42)} = 1680$$

Now, we get that the circumradius satisfies that:

$$[ABC](4R) = (78)(50)(112)$$

$$R = \frac{(78)(50)(112)}{4(1680)} = 65$$

Therefore, the circumradius is 65. Now, we can notice the following. Now, let H be the orthocenter of $\triangle ABC$. Now, let L_1, L_2, L_3 lie on MP, QX, NY respectively. Now, we can notice that L_1, L_2, L_3 are on the triangles of CMP, AQX , and BNY respectively. Now, I claim that L_1, L_2, L_3 lie on CH, AH, BH respectively. We'll prove simply that L_2 lies on AH , and we can apply a similar proof for each of the others. Let AD be an altitude. Consider the rotations of $\triangle ADB, ADC$ around A so that AB, AC match up to AY, AN respectively. Let the points that D gets rotated to be called U, V for triangles ADB, ADC respectively. notice that UA, VA are perpendicular to AD since it's a 90 degree rotation. Therefore, this tells us that UV contains A . Furthermore, notice that UY and VN are both perpendicular to UV , so therefore they're parallel. Furthermore, we can notice that $UA = VA = AD$. Therefore, we get that $UVNY$ is a trapezoid, with A, L_2 as midpoints of their respective points. Therefore, it has to be that AL_2 is perpendicular to UV . Thus, we get that AL_2 is the extension of the altitude AD . We get that L_2A, L_1C, L_3B all intersect at the orthocenter H . Furthermore, by the trapezoidal observation, we get that $|AL_2| = \frac{BC}{2}$. Therefore, from here on we length bash. We want to find the areas of the triangles, $\triangle HL_1L_2, \triangle HL_2L_3, \triangle HL_1L_3$. Let a, b, c be the triangle's angles at points A, B, C respectively. We get that:

$$HL_1 = HC + CL_1 = 2R\cos(c) + R\sin(c)$$

$$HL_2 = HA + AL_2 = 2R\cos(a) + R\sin(a)$$

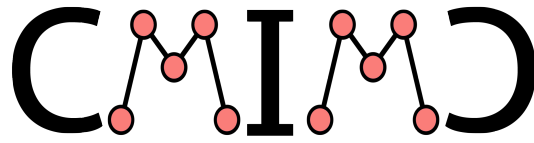
$$HL_3 = 2R\cos(b) + R\sin(b)$$

From here, we get that:

$$HL_1 = 143, HL_2 = 145, HL_3 = (-10)$$

by bashing out the values of sin and kind of directing the values. Then, we get that the angles between HL_2, HL_1 is $180 - b$, which has a sine value of $\sin(b)$. Therefore, we get that the area of

$$\triangle HL_1L_2 = (143)(145)(56/130) = 8932$$



$$\triangle HL_2 L_3 = ((-10))(145)(39/130) = -435$$

$$\triangle HL_1 L_3 = (143)((-10))(25/130) = -275$$

Summing these three areas gives us an area of 8222 as our answer.

10. In a 2024×2024 grid of squares, each square is colored either black or white. An ant starts at some black square in the grid and starts walking parallel to the sides of the grid. During this walk, it can choose (not required) to turn 90° clockwise or counterclockwise if it is currently on a black square, otherwise it must continue walking in the same direction.

A coloring of the grid is called *simple* if it is **not** possible for the ant to arrive back at its starting location after some time. How many simple colorings of the grid are maximal, in the sense that adding any black square results in a coloring that is not simple?

Proposed by James Yang

Answer. 2024^{4046}

Solution. TLDR: Bijection spam :)

We begin by interpreting each row and column of the grid as vertices in a graph. In particular, construct vertices $r_1, r_2, \dots, r_{2024}$, $c_1, c_2, \dots, c_{2024}$, and connect (r_x, c_y) iff the square (x, y) in the grid is black. In this way, every possible coloring of the square grid uniquely corresponds to some bipartite graph on 2 sets of 2024 vertices.

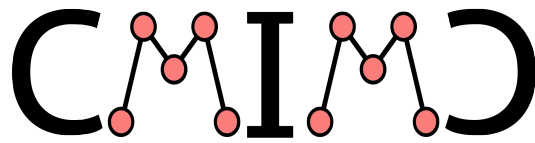
Next, we notice that a coloring of the grid is simple exactly when the corresponding bipartite graph is acyclic. This is because any path the ant takes corresponds to a path in the bipartite graph, as each time the ant switches between rows and columns of the grid, it must be on a black square, hence the corresponding sequence of vertices in the graph is a path. Therefore, it is only possible for the ant to revisit its starting location if the bipartite graph contains a cycle.

Now, we attempt to interpret when a simple coloring is maximal. We know that simple colorings correspond to acyclic bipartite graphs. If there were a pair (r_x, c_y) of vertices in distinct connected components of the graph, then we could simply add this edge and the graph remains acyclic. Thus, any pair (r_x, c_y) is connected by a path. It clearly follows that in fact, any two pair of vertices in the graph are connected by a path. This means that maximal simple colorings exactly correspond to bipartite acyclic connected graphs, or in other words, spanning trees of a $K_{2024, 2024}$.

It then suffices to count the number of distinct spanning trees of a labelled $K_{2024, 2024}$. To do this, we can modify the well-known bijective proof of Cayley's Formula.

First, note that in any bipartite graph, the sum of the degrees of vertices on both sides are equal. Since acyclic graphs have an average degree of < 2 , then any spanning tree of a $K_{m, n}$ has a leaf on the side with more vertices (and both sides if $m = n$). This observation motivates the following algorithm: Given a spanning tree of a labelled $K_{2024, 2024}$, we alternate between sides of the graph (starting from a fixed side), and remove the current lowest index leaf on this side each iteration. We continue this until just 2 vertices remain, and in each iteration, we record in a sequence the unique vertex of the graph adjacent to the removed leaf. This algorithm results in a sequence of 4046 vertices, alternating between sides of the graph.

We show that this algorithm produces a bijection from spanning trees of the labelled $K_{2024, 2024}$ to sequences of 4046 vertices that alternate sides. To do this, we construct the inverse algorithm



to recover the original spanning tree: Given such a sequence of 4046 vertices, collect the set of vertices that do not appear in the sequence, and mark them as "available". For each vertex in the sequence in order, attach it to the lowest index 'available' vertex on the other side. The previously available vertex that was attached to is now no longer available, whereas the vertex from the sequence, if it is new, is now available. After finishing this process, there will just be one available vertex left on each side, and connected these two vertices reconstructs the desired spanning tree of the $K_{2024,2024}$.

Through this above bijection, we see that the answer to the problem is just the number of possible sequences of 4046 vertices that alternate sides starting from a fixed side. This is just 2024^{4046} .

Remarks: The general formula for the number of spanning trees in a labelled $K_{m,n}$ is known (although hopefully not by students) to be $m^{n-1}n^{m-1}$. This problem can also be solved using exponential generating functions or Kirchhoff's matrix tree theorem.

11. **(Tiebreaker)** Give us a 8-character string (of letters and numbers) and tie is broken by the highest CRC-16 hash value.

```
# from https://stackoverflow.com/questions/10564491/function-to-calculate-
# a-crc16-checksum
def crc16(string):
    crc = 0
    for char in string:
        crc ^= ord(char)
        for _ in range(8):
            isOdd = crc % 2 == 1
            crc //= 2
            if isOdd:
                crc ^= 0xa001
    return crc
```

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