2025 CMIMC Integration Bee

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Rules

1. You have **50 minutes** to solve the following **15 problems**, which are roughly ordered by difficulty. Note that later problems are worth more points.

2. Write your answers on the answer sheets provided. You don't need to put your name on it, but make sure to include the number of the question you're answering. When you're ready to submit, raise your hand up with the answer sheet and one of the runners will come pick it up. You get a total of **20 total submissions**, and there's no limit on the number of submissions for a single problem. Your packet contains exactly 20 answer sheets (which should be enough), but if you happen to need replacements, let one of the staff members know.

3. During the test, there will be a live leaderboard with everyone's score (if you don't see your name on the leaderboard at this time, please let us know). As the answers are graded, be aware that there will be a delay until you see feedback on the leaderboard. However, if you believe that one of your submitted answer sheets was not processed (after a reasonable amount of time), let one of the staff members know.

4. Submit your answers in simplest form, expressed in terms of standard mathematical functions. Please write neatly: if your answers are indecipherable, we will likely just mark them as wrong. Specifically, we will assume that ln and log both refer to the natural logarithm. Otherwise, please specify the base.

5. The following notation may be useful:
   - $\log(x)$ denotes the natural (base-e) logarithm of x.
   - $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. $\{x\}$ denotes the fractional part of x, i.e. $x - \lfloor x \rfloor$. $\lceil x \rceil$ denotes the least integer greater than or equal to x.
   - $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.
   - for $x > 0$, $\lambda(x)$ denotes the unique positive real root of $t^3 - xt - 1 = 0$.
   - for $0 < x < 1$ and integer n, $\mathrm{bi}_n(x)$ denotes the number of ones in the first n digits after the decimal point in the binary expansion of x.

6. No computational aids other than a writing utensil and blank paper may be used (if you require additional scratch paper at any point during the test, let one of the staff members know). In particular, **no electronic aids** may be used during the test. Any competitor caught using a computational device during the duration of the test will be immediately disqualified.

7. In the event of a tie between top scorers, all involved participants will be called up for an estimation-style tiebreaker at the end of the written portion. Ties will be broken by the closeness of the submitted answer to the exact value of the integral.

Additional thanks to Connor Gordon for proposing problems for the test!

Problems

1. (5 points)

$$\int_{-\pi}^{\pi} \sin(x) \left(\frac{\sec(x) + \csc(x)}{x^{-2}} \right) \mathrm{d}x$$

2. (5 points)

$$\int_{1}^{4} \prod_{i=0}^{\infty} \sqrt[2^i]{x} \, \mathrm{d}x$$

3. (5 points)

$$\int_{e^e}^{e^{e^e}} \frac{1}{\log_{\log(x)}(x^x)} \, \mathrm{d}x$$

4. (6 points)

$$\int_{0}^{1} \mathrm{bi}_{2025}(x) \, \mathrm{d}x$$

5. (6 points)

$$\int_{\pi/6}^{\pi/2} \frac{\sin(3x)}{\sin^3(x)} + \frac{\cos(3x)}{\cos^3(x)} \, \mathrm{d}x$$

6. (6 points)

$$\int_{-2025}^{2025} \left\lceil x - \frac{1}{2} \right\rceil \cdot \left\lceil \frac{1}{x} - \frac{1}{2} \right\rceil \, \mathrm{d}x$$

7. (7 points)

$$\int_{0}^{1} 4^{\sqrt[4]{x}-1} \, \mathrm{d}x$$

8. (7 points)

$$\int_{1}^{1+\varphi} (x^2 - 1) \sum_{k=0}^{100} \binom{100}{k} x^{2k-102} \, \mathrm{d}x$$

9. (7 points)

$$\int_{0}^{1} \sum_{n=0}^{\infty} \left(\frac{3}{4} \right)^{n} \left| 2\{4^n x\} - 1 \right| \mathrm{d}x$$

10. (8 points)

$$\int_{0}^{\infty} \sum_{k=1}^{2025} \frac{1}{(k^2 + k)x^2 + (2k+1)x + 1} \, \mathrm{d}x$$

11. (8 points)

$$\int_{0}^{2} \frac{x}{3\lambda(x)^2 - x} \, \mathrm{d}x$$

12. (9 points)

$$\int_{0}^{1} \frac{x^2 - 2\log(x) - 1}{\log^2(x)} \, \mathrm{d}x$$

13. (9 points)

$$\int_{0}^{1/\sqrt{3}} \frac{x}{1 - x^4} \left[\log(1 + x^2) - \log(1 - x^2) \right] \mathrm{d}x$$

14. (10 points)

$$\int_{-\infty}^{\infty} 1 - \cos\left(\frac{1}{x-1} + \frac{1}{x+1} \right) \mathrm{d}x$$

15. (10 points)

$$\int_{0}^{\pi/2} \tan(x)^{4/5} \, \mathrm{d}x$$