

Geometry Round Solutions

1. I'm given a square of side length 7, and I want to make a regular tetrahedron from it. Specifically, my strategy is to cut out a net. If I cut out a parallelogram-shaped net that yields the biggest regular tetrahedron, what is the surface area of the resulting tetrahedron?

Proposed by Lohith Tummala

Answer. $14\sqrt{3}$

Solution.

We can maximize the size if we place the far opposite corners of the parallelogram on opposite corners of the square. Call the side length of the tetrahedron (the short side of the parallelogram) x . The long side is $2x$, and via either Law of Cosines or Pythagorean Theorem, we get the long diagonal (also the diagonal of the square) to be $x\sqrt{7}$. The actual length of the square's diagonal is $7\sqrt{2}$, so we can solve for x to be $\sqrt{14}$. To get the surface area, we can take four times the area of each of the four equilateral triangles in the parallelogram.

$$4 \cdot \frac{(\sqrt{14})^2 \sqrt{3}}{4} = \boxed{14\sqrt{3}}.$$

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2. Given a cube of side length 4, place eight spheres of radius 1 inside the cube so that each sphere is externally tangent to three others. What is the radius of the largest sphere contained inside the cube which is externally tangent to all eight?

Proposed by Jenny Quan

Answer. $\sqrt{3} - 1$

Solution.

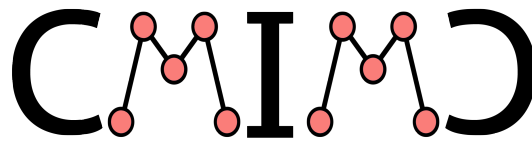
Connecting the centers of the spheres in the corner creates a cube of side length 2. The distance from one corner of this cube to the opposite corner is $2\sqrt{3}$. Halving this value gives us the radius of the central sphere plus the radius of a corner sphere, so the radius of the central sphere is $\boxed{\sqrt{3} - 1}$.

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3. Let AB be a segment of length 1. Let $\odot A, \odot B$ be circles with radius $\overline{AB}$ centered at A, B . Denote their intersection points C, D . Draw circles $\odot C, \odot D$ with radius $\overline{CD}$. Denote the intersection points of $\odot C$ and $\odot D$ by E, F . Draw circles $\odot E, \odot F$ with radius $\overline{EF}$ and denote their intersection points G, H .

Compute the area of the pentagon $ACFHE$.

Proposed by Robert Trosten

Answer. $\frac{11\sqrt{3}}{4}$



Solution. Call the midpoint of AB as O . We see that $\triangle ACO$ is a $30 - 60 - 90$ triangle. This means that $CO = \frac{\sqrt{3}}{2}$ and thus, $CD = \sqrt{3}$. We can also conclude that $EF = 3$ and $GH = 3\sqrt{3}$. (To see this, notice that we just rotate circles A and B about O by 90° and scale by $\sqrt{3}$ to get circles C and D . Perform this process on circles C and D to get circles E and F . Perform this process on circles E and F to get circles G and H .)

Draw line EF . We aim to calculate the areas of $\triangle EHF$ and $\triangle ACF$ separately.

$$[\triangle EHF] = \frac{1}{2} \cdot 3 \cdot \frac{3\sqrt{3}}{2} = \frac{9\sqrt{3}}{4}$$

$$[\triangle ACF] = \frac{1}{2} \cdot 2 \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$$

Add these two to get $\boxed{\frac{11\sqrt{3}}{4}}$.

4. Let $ABCDEF$ be a regular hexagon with side length 1, and let G be the midpoint of side $\overline{CD}$, and define H to be the unique point on side $\overline{DE}$ such that $AGHF$ is a trapezoid. Find the length of the altitude dropped from point H to $\overline{AG}$.

Proposed by Michael Duncan

Answer. $\frac{2\sqrt{39}}{13}$

Solution.

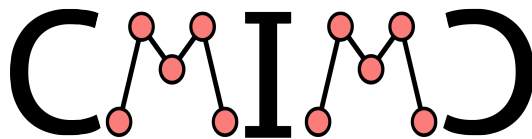
For $AGHF$ to be a trapezoid, it must have a pair of opposite parallel sides. Either AF and GH are parallel, or AG and EF are parallel. It is clear that AF and GH cannot be parallel, as that would make H be outside the hexagon, hence $AG \parallel HF$.

Now, it is relatively easy to find the length of AG , so if we can find the area of $\triangle AHG$, we would get the altitude quickly from the formula $\frac{1}{2}bh$. Hence, to find the area of the triangle, since $AG \parallel HF$, we can "slide" point H to F without changing the area of the triangle. We can then go a step further, and slide point G to D (although this doesn't change the proceeding calculation). To find the height of the altitude from G to side length AF , note that since we can partition the hexagon into 6 equilateral triangles, each having height $\frac{\sqrt{3}}{2}$, the height must be double this, $\sqrt{3}$. Thus, we find that the area of $\triangle AGH = \frac{\sqrt{3}}{2}$.

Now, by Pythagorean's Theorem, we find that $AG = \frac{\sqrt{13}}{2}$, and thus for our desired altitude h ,

$$\frac{1}{2} \cdot \frac{\sqrt{13}}{2} \cdot h = \frac{\sqrt{3}}{2}$$

Solving yields $h = \boxed{\frac{2\sqrt{39}}{13}}$, as desired.



5. Let $\triangle ABC$ be an equilateral triangle. Let E_{AB} be the ellipse with foci A, B passing through C , and in the parallel manner define E_{BC}, E_{AC} . Let $\triangle GHI$ be a (nondegenerate) triangle with vertices where two ellipses intersect such that the edges of $\triangle GHI$ do not intersect those of $\triangle ABC$. Compute the ratio of the largest sides of $\triangle GHI$ and $\triangle ABC$.

Proposed by Robert Trosten

Answer. $\frac{12\sqrt{3} + 4}{13}$

Solution.

Let $AB = BC = AC = 1$. Let's look at E_{AB} . Let the intersection point closer to A be G . Let the intersection point closer to B be H .

I claim that $\angle GAB = 150^\circ$. From symmetry, GHI is equilateral. Ellipse E_{AB} has two intersection points G and H , and GH is parallel to AB . Thus, $GABH$ is a trapezoid, and the claim follows.

Because we have an ellipse, $GA + AB = AC + CB = 2$. Thus, let's assign $GA = x$ and $GB = 2 - x$.

We can use Law of Cosines to get $x = \frac{3}{4 + \sqrt{3}}$.

Now, we want the horizontal distance from G to H . To do this, we need to drop altitudes from G to AB and H to AB , and call the foots G' and H' . It suffices to find $G'H'$, which is the same as $G'A + AB + BH'$.

$$G'A = \frac{3}{4 + \sqrt{3}} \cos(30^\circ) = \frac{3\sqrt{3}}{8 + 2\sqrt{3}}$$

$$BH' = G'A$$

via symmetry.

$$G'A + AB + BH' = \frac{3\sqrt{3}}{8 + 2\sqrt{3}} + 1 + \frac{3\sqrt{3}}{8 + 2\sqrt{3}} = \boxed{\frac{12\sqrt{3} + 4}{13}}.$$

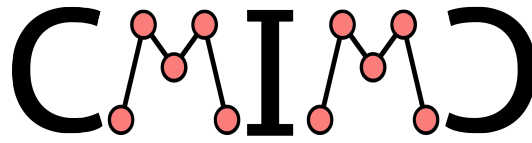
6. Points A, B, C, D, E , and F lie on a sphere with center O and radius R such that $\overline{AB}, \overline{CD}$, and $\overline{EF}$ are pairwise perpendicular and all meet at a point X inside the sphere. If $AX = 1$, $CX = \sqrt{2}$, $EX = 2$, and $OX = \frac{\sqrt{2}}{2}$, compute the sum of all possible values of R^2 .

Proposed by Connor Gordon

Answer. $\frac{31}{7}$

Solution. Note that A, B, C , and D are coplanar, so they lie on the intersection of the sphere with a plane, which is a circle, thus by power of a point $AX \cdot BX = CX \cdot DX$. Similar reasoning gives this quantity (call it k) is also equal to $EX \cdot FX$.

Construct a cube with opposite vertices O and X and edges perpendicular to $\overline{AB}, \overline{CD}$, and $\overline{EF}$. Dropping a perpendicular from O to the plane containing A, B, C , and D , we move from a point in line with the midpoint of $\overline{EF}$ to a point in the plane. Letting $EX = e$ and $FX = f$ for brevity, this means the length of this perpendicular (which is also an edge of the cube) is $\frac{e+f}{2} - \min(e, f) = \frac{|e-f|}{2}$. Similar reasoning gives the other two side lengths to be $\frac{|a-b|}{2}$ and $\frac{|c-d|}{2}$.



We can then compute $OX^2 = (\frac{a-b}{2})^2 + (\frac{c-d}{2})^2 + (\frac{e-f}{2})^2 = \frac{a^2+b^2+c^2+d^2+e^2+f^2-6k}{4}$. Moreover, by our above power of a point argument, $b = \frac{k}{a}$, $d = \frac{k}{c}$, and $f = \frac{k}{e}$, so we can substitute in the given values to get $\frac{1}{2} = \frac{1+k^2+2+\frac{k^2}{2}+4+\frac{k^2}{4}-6k}{4}$, or $7k^2 - 24k + 28 = 8$. Rearranging and factoring gives $(k-2)(7k-10)$, so $k = 2$ or $k = \frac{10}{7}$.

Now, drawing in OA , we can do a similar "space diagonal of a cube" argument to get that $R^2 = (\frac{a+b}{2})^2 + (\frac{c-d}{2})^2 + (\frac{e-f}{2})^2 = \frac{a^2+b^2+c^2+d^2+e^2+f^2-2k}{4} = OX^2 + k = \frac{1}{2} + k$. Plugging in our two values of k gives $\frac{1}{2} + 2 = \frac{5}{2}$ and $\frac{1}{2} + \frac{10}{7} = \frac{27}{14}$ respectively, for a sum of $\boxed{\frac{31}{7}}$.

7. Let ABC be a triangle with altitude $\overline{AF}$. Let $AB = 5, AC = 8, BC = 7$. Let P be on $\overline{AF}$ such that it lies between A and F . Let ω_1, ω_2 be the circumcircles of APB, APC respectively. Let $\overline{BC}$ intersect ω_1 at $B' \neq B$. Also, let $\overline{BC}$ intersect ω_2 at $C' \neq C$. Let $X \neq A$ be on ω_1 such that $B'X = B'A$. Let $Y \neq A$ be on ω_2 such that $C'A = C'Y$. Let X, Y, A all lie on one line h . Find the length of PA .

Proposed by Henry Zheng

Answer. $\frac{7\sqrt{3}}{3}$

Solution. Notice that $\angle C'AY = \angle ACC' = \angle ACB$ by the length of arcs since we know that $C'A = C'Y$. By the same logic, we get that $\angle B'AX = \angle ABC$. Extend YC and XB to intersect at a point K . We know that by cyclic quadrilaterals, that $\angle BCK = \angle C'AY = \angle ACB$ by the fact that opposite angles sum to 180. Similarly, we get that $\angle CBK = \angle XAB' = \angle ABC$. Therefore, we get that K is just A reflected over BC , as the angles are equal. Then, we can notice that AF intersects K , so therefore AK is a radical axis and B, C, X, Y are cyclic. From there, by length bashing, we get that the area of ABC is by heron's:

$$\sqrt{10(5)(2)(3)} = 10\sqrt{3}$$

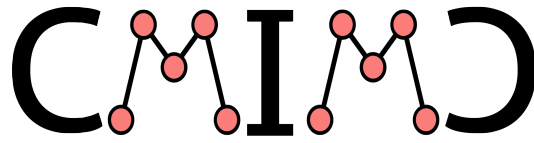
which gives us the height of $AF = KF$ has $\frac{20\sqrt{3}}{7}$. Now, we can notice that since B, C, X, Y are cyclic, and A is on line XY , we get that $\angle KAX = \angle KBP$ and that angle $\angle KAY = \angle KCP$. Since their sum is 180 degrees, it must be that K, B, C, P must be cyclic. Then, by power of a point, we know that $CF \cdot BF = KF \cdot PF$. Doing the calculation yields that $CF = \frac{44}{7}, BF = \frac{5}{7}$, and that

$$\frac{220}{49} = \frac{20\sqrt{3}}{7} \cdot PF$$

This gives us that $PF = \frac{11\sqrt{3}}{21}$, which gives us that $AP = \boxed{\frac{49\sqrt{3}}{21}} = \boxed{\frac{7\sqrt{3}}{3}}$.

8. Let ω be a circle with diameter $\overline{AB}$, center O , and cyclic quadrilateral $ABCD$ inscribed in it, with C and D on the same side of $\overline{AB}$. Let $AB = 20, BC = 13, AD = 7$. Let $\overleftrightarrow{BC}$ and $\overleftrightarrow{AD}$ intersect at E . Let the E -excircle of ECD have its center at L . Find OL .

Proposed by Henry Zheng



Answer. $\frac{97 - 3\sqrt{1001}}{2}$

Solution. Let J be the point on AB such that BJ = 13, AJ = 7. Consider the circumcircle of CDJ, and let it intersect AB again at X. Let point $\angle DAB = 180 - 2a$, $\angle CBA = 180 - 2b$. Then, we can note that by angle chasing:

$$\angle CJB = \angle JCB = \angle CDX = b = \angle CDA/2$$

Now, we can notice that DX bisects the angle $\angle CDA$. Similarly, CX must bisect that angle as well. Therefore, X is actually our desired point L, and it lies on AB. Now, it suffices to find where the angle bisector of $\angle CED$ intersects AB because that is L. We can get that:

$$\sin(\angle DAB) = \sqrt{100 - 3.5^2}/10 = \sqrt{351}/20$$

$$\sin(\angle CBA) = \sqrt{100 - 6.5^2}/10 = \sqrt{231}/20$$

Then, taking their ratios and subtracting the length, we get that the answer should be:

$$10 - \frac{20\sqrt{231}}{\sqrt{351} - \sqrt{231}}$$

Simplifying it down, we get the answer:

$$OL = \boxed{\frac{97 - 3\sqrt{1001}}{2}}.$$

9. Define the *ratio* of an ellipse to be the length of the major axis divided by the length of the minor axis.

Given a trapezoid $ABCD$ with $AB \parallel DC$ and that $\angle ADC$ is a right angle, with $AB = 18$, $AD = 33$, $CD = 130$, find the smallest ratio of any ellipse that goes through all vertices of $ABCD$.

Proposed by Ishin Shah

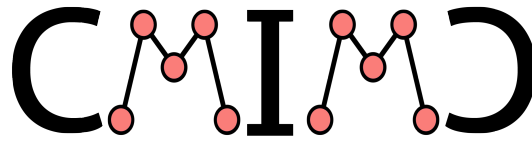
Answer. $\frac{11}{3}$

Solution. Note that an ellipse is some scaled up version of a circle, so we want to minimize the scaling factor.

Thus, we want to scale $ABCD$ down to some $A'B'C'D'$ in some direction such that $A'B'C'D'$ is cyclic.

Under scaling, parallel lines stay parallel, so $A'B' \parallel D'C'$. Thus, we have $A'B'C'D'$ is a cyclic trapezoid making it isosceles. Thus, our condition we need is $A'D' = B'C'$.

Furthermore, if we changed AB and CD by the same amount, this is the same as sliding BC with B on AB and C on CD . Therefore, we could change our objective to finding the scaling factor of a right triangle XYZ with legs $XY = 33$ and $YZ = 112$ to make $X'Y' = X'Z'$, as XY corresponds to AD and XZ corresponds to BC . Let M be the midpoint of YZ . Then under our scaling, since $Y'Z' = X'Z'$, we need M' to be orthogonal to $Y'Z'$.



We could scale our triangle from any center, so we will center it at M . Let ℓ be the line we scale onto and let θ be the angle between ℓ and YM , which makes the angle between ℓ and XM equal to γ , with our scaled down angles being θ', γ' . We want $\tan(\theta') \tan(\gamma') = -1$, and if we scale by a factor of k , then we get $\tan(\theta') = \tan(\theta)k, \tan(\gamma') = \tan(\gamma)k$.

Thus, we need to find the smallest k such that $\tan(\theta) \tan(\gamma) = \frac{-1}{k^2}$ has a solution.

Note that $\gamma = \theta + \tan^{-1}(\frac{33}{56})$ so if $\tan(\theta) = t$, our equality is now $t(\frac{t + \frac{33}{56}}{1 - \frac{33}{56}t}) = -\frac{1}{k^2}$. Let

$x = 1 - \frac{33}{56}t$, making this simplify to $-(2(\frac{56}{33})^2 + 1) + \frac{1}{33^2}(56^2x + \frac{65^2}{x})$. This is minimized with $x = \frac{65}{56}$ by AM-GM, so our answer is $-(2(\frac{56}{33})^2 + 1) + \frac{2 * 56 * 65}{33^2} = -(\frac{56^2}{33} + \frac{65^2}{33} - \frac{2 * 56 * 65}{33^2}) = -(\frac{(65 - 56)^2}{33^2}) = -(\frac{3}{11})^2$.

This makes $-\frac{1}{k^2} = -(\frac{3}{11})^2$ so $k = \boxed{\frac{11}{3}}$.

10. Let $\triangle ABC$ exist such that $AB = 6, BC = 8, AC = 10$. Let P lie on the circumcircle of ABC , ω , such that P lies strictly on the arc in between B and C (i.e. $P \neq B, C$). Drop altitudes from P to BC, AC at points J and Q respectively. Let l be a line through B such that it intersects AC at a point K . Let M be the midpoint of BQ . Let CM intersect line l at a point I . Let AI intersect JQ at a point U . Now, B, J, U, M are cyclic. Now, let $\angle QJC = \theta$. If we set $y = \sin(\theta), x = \cos(\theta)$, they satisfy the equation:

$$768(xy) = (16 - 8x^2 + 6xy)(x^2y^2(8x - 6y)^2 + (8x - 8xy^2 + 6y^3)^2)$$

The numeric values of s, c are approximately:

$$x = 0.72951, y = 0.68400$$

Let BK intersect the circumcircle of ABC , ω at a point L . Find the value of BL . We will only look up to two decimal places for correctness.

Proposed by Henry Zheng

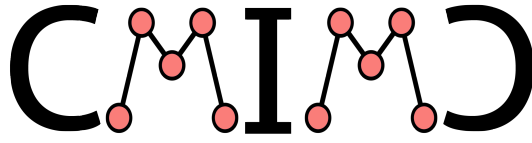
Answer. 9.94

Solution.

We will set the values $x = \cos(\theta), y = \sin(\theta)$ with $\theta = \angle QJC$ as stated in the problem. First, notice that JQ is the Simpson line. Therefore, let JQ intersect AB at a point X . Let PQ intersect the circle at a point L' . Draw BL' . Now, I claim that BL' is parallel to the Simpson line of JQ .

Notice that $\angle PCB = \widehat{PB}/2$. However, we can notice that since P, J, Q, C are cyclic because $\angle PJC = \angle PQC = 90^\circ$. Then, we get that $\angle PCB = \angle PCJ = \angle PQJ$. However, we also know that $\angle PLB = \widehat{PB}/2$ by arc lengths so therefore it has to be that $\angle PLB = \angle PQJ$, which means that $BL' \parallel JQ$.

Notice that by some trig bashing, that the length of BL' is exactly $8x + 6y$. Similarly, by doing the opposite trig bash, we get that $BP = 8x - 6y$. Now, notice that $BXPJ$ is a rectangle, and



that the angle of its diagonal is also $\angle CJQ = \theta$. Now, notice that BP is the reflection of BL' over BC . Therefore, by another quick angle bash, we get that the length of BP is just $8x - 6y$. Now, let JX intersect BP at Y . Notice that QY bisects BP because it's a rectangle. Thus, we would get that $QY = \frac{BL'}{2}$. From here, we can get that the length of XQ is just:

$$XQ = BY + YQ = 0.5(BL' + BP) = 8x$$

Furthermore, since $XJ = BP = 8x - 6y$, we get that $JQ = 6y$. Now, we know that CM bisects BQ . Now, let CM intersect JQ at Z . We will now do mass points on $\triangle BQC$. We know that by a bit more trig bashing that:

$$BJ = (8x - 6y)x$$

Therefore, if we let $BJ = a$, we can notice that the following weight are true (each weight of the vertex corresponds to its letter):

$$w_B = w_q$$

$$w_B(8 - a) = w_c(a)$$

Therefore, we can model this with the weight at J is 8 and the weight at Q is $8 - a$. Thus, the ratio of $JZ : ZQ$ is equal to $8 - a : 8$. Therefore, we get that:

$$JQ = \frac{48y}{16 - (8x - 6y)x}, BQ = 8x$$

Now, we can notice the following. The altitude from B to JQ , which we'll call BH has length equal to:

$$BH = (8x - 6y)xy$$

Furthermore, we can get that:

$$XH = (8x - 6y)y^2$$

Therefore, we can get the following:

$$BQ^2 = HQ^2 + BH^2$$

$$BQ^2 = (XQ - XH)^2 + BH^2$$

$$BQ^2 = (8x - (8x - 6y)y^2)^2 + (8x - 6y)^2 x^2 y^2$$

$$BQ^2 = (8x - 8xy^2 + 6y^3)^2 + (8x - 6y)x^2 y^2 = 2(QU)(QB)$$

by power of a point, as B, J, U, M are cyclic and the fact that M bisects BQ . However, we can also notice that:

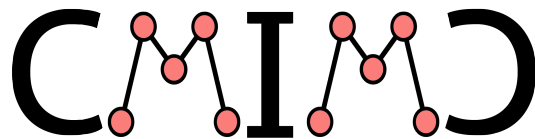
$$(QZ)(QX) = \frac{48y}{16 - 8x^2 + 6xy}(8x) = \frac{384xy}{16 - 8x^2 + 6xy}$$

However, we can notice that from the handy dandy fact that was given to us in the problem by what θ is, we get that:

$$(QZ)(QX) = \frac{(16 - 8x^2 + 6xy)(x^2 y^2 (8x - 6y)^2 + (8x - 8xy^2 + 6y^3)^2)}{2((16 - 8x^2 + 6xy))}$$

$$(QZ)(QX) = \frac{1}{2}(x^2 y^2 (8x - 6y)^2 + (8x - 8xy^2 + 6y^3)^2) = \frac{1}{2}BQ^2$$

$$(QZ)(QX) = (QU)(QB)$$



Now, notice that since CM, AM intersect JQ at points Z, U and AB, BC intersect JQ at points J, X , we can apply Dual Desargues Involution Theorem on the degenerate quadrilateral $ABIC$. This gets us that K is the center of the desired involution by DDIT on the line JQ . However, K has to send the point where JQ intersects AC to where it intersects BI . However, since K is on AC , it has to send to the point at infinity after such an involution, meaning that $BI = l$ is parallel to JQ . However, that tells us that $L' = L$, so we just need to find the length of BL' , which we know is:

$$BL' = 8x + 6y = \boxed{9.94}.$$

11. **(Tiebreaker)** We have a point P , and points $A_0, A_1, \dots$ that such that they are all distance 2 from P , they are counter clockwise around P , and there exists a circle of radius $\frac{1}{2^i}$ tangent to PA_i and PA_{i+1} at A_i and A_{i+1} . Find the limit of the angle $\angle A_0PA_n$ as n goes to ∞ (in radians). Express it as $a.bcd\text{efg}$ (round to 6 digits)

Proposed by Henry Zheng

Answer. 1.91578

Solution. This was done by graphing to estimate the value.