Combinatorics and Computer Science Round Solutions

1. Robert has five beads in his hand, with the letters C, M, I, M, and C, and he wants to make a circular bracelet spelling "CMIMC." However, the power went out, so Robert can no longer see the beads in his hand. Thus, he puts the five beads on the bracelet randomly, hoping that the bracelet, when possibly rotated or flipped, spells out "CMIMC." What is the probability that this happens? (Robert doesn't care whether some letters appear upside down or backwards.)

   Proposed by Lohith Tummala

   Answer. $\frac{1}{6}$

   Solution. There are $\frac{5!}{2!2!} = 30$ ways to arrange these beads in a line, assuming that the C and M bead pairs are indistinguishable. Out of these ways, there are exactly five of them that can be rotated to get CMIMC: CMIMC, MIMCC, IMCCM, MCCMI, CCMIM Thus, since each of the thirty ways are equally likely, we get $\frac{5}{30} = \boxed{\frac{1}{6}}$

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2. Every day, Pinky the flamingo eats either 1 or 2 shrimp, each with equal probability. Once Pinky has consumed 10 or more shrimp in total, its skin will turn pink. Once Pinky has consumed 11 or more shrimp in total, it will get sick. What is the probability that Pinky does not get sick on the day its skin turns pink?

   Proposed by Connor Gordon

   Answer. $\frac{683}{1024}$

   Solution. Let $P(x)$ denote the probability of success (Pinky turns pink but doesn't get sick) assuming that Pinky has eaten x shrimp already. Note that $P(11) = 0$ since Pinky would be sick. Also, $P(10) = 1$ since Pinky's skin is pink but Pinky is not sick. After that, since Pinky can eat either one or two shrimp, each with half chance,

$$P(x) = \frac{1}{2}(P(x+1) + P(x+2))$$

   We can then compute the probabilities all the way to $P(0)$.

$$P(9) = \frac{1}{2}, \quad P(8) = \frac{3}{4}, \quad P(7) = \frac{5}{8}, \quad P(6) = \frac{11}{16}, \cdots$$

   We notice that the denominator keeps doubling, while the numerator doubles and flips between adding one or subtracting one. We eventually get our answer of $\boxed{\frac{683}{1024}}$.

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3. There are 34 friends are sitting in a circle playing the following game. Every round, four of them are chosen at random, and have a rap battle. The winner of the rap battle stays in the circle and the other three leave. This continues until one player remains. Everyone has equal rapping ability, i.e. every person has equal probability to win a round. What is the probability that Michael and James end up battling in the same round?

Proposed by Michael Duncan

Answer. $\dfrac{2}{17}$

Solution. The game lasts 11 rounds. By symmetry, the ith round of a game is equally likely to be any of the $\binom{34}{4}$ groups. There are $\binom{32}{2}$ valid groups containing Michael and James. So the probability is just $11 \cdot \binom{32}{2}/\binom{34}{4} = \boxed{\dfrac{2}{17}}$.

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4. Let n and k be positive integers, with $k \leq n$. Define a (simple, undirected) graph $G_{n,k}$ as follows: its vertices are all of the binary strings of length n, and there is an edge between two strings if and only if they differ in exactly k positions. If $c_{n,k}$ denotes the number of connected components of $G_{n,k}$, compute

$$\sum_{n=1}^{10} \sum_{k=1}^{n} c_{n,k}.$$

(For example, $G_{3,2}$ has two connected components.)

Proposed by Robert Trosten and Allen Yang

Answer. 1088

Solution. Let $n \geq 1$.

First observe the edge case: $c_{n,n} = 2^{n-1}$, as every string is only paired up with the string that differs in all positions.

For $0 < k < n$ a parity argument will demonstrate $c_{n,k} = 1$ for k odd and $c_{n,k} = 2$ for k even. Handling the sum yields the final result which is $\boxed{1088}$.

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5. Consider a 12-card deck containing all four suits of 2, 3, and 4. A *double* is defined as two cards directly next to each other in the deck, with the same value. Suppose we scan the deck left to right, and whenever we encounter a double, we remove all the cards up to that point (including the double). Let N denote the number of times we have to remove cards. What is the expected value of N?

Proposed by Michael Duncan

Answer. $\dfrac{138}{55}$

Solution. We can use linearity of expectation to compute the number of doubles in the deck. There are 11 pairs of consecutive cards. In each pair, the second card is equally likely to be any of the 11 cards other than the first card. Of these 11 cards, 3 of them match the first card. Thus the expected number of doubles is $11 \cdot \frac{3}{11} = 3$.

However, if there is a group of three of the same card in a row, removing the first double will only leave one card. And if all four cards of the same value are in a row, we remove cards twice. Therefore we also have to track the number of triples and quadruples in the deck.

If there are two of the same value in a row, there is one double, and we remove cards once. If there are three of the same value in a row, there are 2 doubles and 1 triple, and we remove cards once. If there are four of the same value in a row, there are 3 doubles, 2 triples, and 1 quadruple, and we remove cards twice. The total number of times we remove cards is

$$N = \#\text{doubles} - \#\text{triples} + \#\text{quadruples}.$$

We can use a similar linearity of expectation argument to find $\mathbb{E}[\#\text{triples}]$ and $\mathbb{E}[\#\text{quadruples}]$. For triples, there are 10 sets of three cards in a row, and the probability that both the second and third card match the first is $\binom{3}{2}/\binom{11}{2} = \frac{3}{55}$. For quadruples, there are 9 sets of 4 cards in a row, and the probability that the second, third, and fourth cards match the first is $\binom{3}{3}/\binom{11}{3} = \frac{1}{165}$. Finally we compute

$$\mathbb{E}[\#\text{doubles}] - \mathbb{E}[\#\text{triples}] + \mathbb{E}[\#\text{quadruples}] = 11 \cdot \frac{3}{11} - 10 \cdot \frac{3}{55} + 9 \cdot \frac{1}{165} = \boxed{\frac{138}{55}}.$$

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6. Consider a 4×4 grid of squares. We place coins in some of the grid squares so that no two coins are orthogonally adjacent, and each 2×2 square in the grid has at least one coin. How many ways are there to place the coins?

Proposed by Justin Hsieh

Answer. 256

Solution. We place coins on each row of the grid, starting from the top. There are a total of eight possible configurations for each row, listed below ($\bullet$ denotes a coin, and $\cdot$ denotes no coin)

- 1: $\cdot\ \cdot\ \cdot\ \cdot$
- 2A: $\bullet\ \cdot\ \cdot\ \cdot$ (and its mirror image, 2B: $\cdot\ \cdot\ \cdot\ \bullet$)
- 3A: $\cdot\ \bullet\ \cdot\ \cdot$ (and 3B: $\cdot\ \cdot\ \bullet\ \cdot$)
- 4A: $\bullet\ \cdot\ \bullet\ \cdot$ (and 4B: $\cdot\ \bullet\ \cdot\ \bullet$)
- 5: $\bullet\ \cdot\ \cdot\ \bullet$

The other configurations violate the condition that no two coins can be adjacent.

Each row can be followed by any other row, so long as both conditions are met. We will list out all possible pairs of rows:

- 1 and (4A or 4B)
- 2A and (3B or 4B)
- 2B and (3A or 4A)
- 3A and (2B or 4A or 4B or 5)
- 3B and (2A or 4A or 4B or 5)
- 4A and (1 or 2B or 3A or 3B)
- 4B and (1 or 2A or 3A or 3B)
- 5 and (3A or 3B)

At this point we can make the following simplification: there are two types of configurations, "center" rows with a coin in either of the center two places (3A, 3B, 4A, 4B), and "non-center" rows without a coin in either of the center two places (1, 2A, 2B, 5). Each center row can be followed by two center rows and two non-center rows. Each non-center row can be followed by two center rows.

Let x_n be the number of configurations consisting of n rows and ending with a center row, and let y_n be the same thing for non-center rows. We have the following recurrences:

$$x_{n+1} = 2x_n + 2y_n, \quad y_{n+1} = 2x_n, \quad x_1 = 4, \quad y_1 = 4.$$

This recurrence gives

$$\begin{aligned} x_2 &= 16, \quad y_2 = 8 \\ x_3 &= 48, \quad y_3 = 32 \\ x_4 &= 160, \quad y_4 = 96. \end{aligned}$$

The total number of valid configurations with 4 rows is $x_4 + y_4 = \boxed{256}$.

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7. Alan is bored one day and decides to write down all the divisors of 1260^2 on a wall. After writing down all of them, he realizes he wrote them on the wrong wall and needs to erase all his work. Every second, he picks a random divisor which is still on the wall and instantly erases it and every number that divides it. What is the expected time it takes for Alan to erase everything on the wall?

   Proposed by Alan Abraham

   Answer. $\dfrac{137^2 \cdot 11^2}{360^2}$

   Solution. Let $n = 1260^2 = 2^4 \cdot 3^4 \cdot 5^2 \cdot 7^2$, and let $\sigma(x)$ denote the number of divisors of x. Instead of picking a random remaining number after each second, it is equivalent for Alan to randomly order the numbers at the beginning and then choose the leftmost number that does not divide any number he has already chosen. With this model in mind, we see that the expected time is equivalent to the expected number of divisors that Alan picks.

   For any divisor $d \mid n$, we want to compute the probability that Alan picks d. Alan will pick d iff every multiple of d (aside from d) lies to the right of it. Since there are $\sigma(n/d)$ multiples of d, this happens with probability $\frac{1}{\sigma(n/d)}$. So by linearity of expectation, the expected number of divisors Alan selects will be

$$\sum_{d|n} \frac{1}{\sigma(n/d)} = \sum_{d|n} \frac{1}{\sigma(d)}$$

   Since $\frac{1}{\sigma(x)}$ is a multiplicative function it suffices to evaluate this function at prime powers. For any prime power p^k we can see

$$\sum_{d|p^k} \frac{1}{\sigma(d)} = 1 + \frac{1}{2} + \cdots + \frac{1}{k+1}.$$

   Hence, our answer is

$$\left(1 + \frac{1}{2} + \cdots + \frac{1}{5}\right)^2 \left(1 + \frac{1}{2} + \frac{1}{3}\right)^2 = \frac{137^2 \cdot 11^2}{360^2}$$

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8. Divide a regular 8960-gon into non-overlapping parallelograms. Suppose that R of these parallelograms are rectangles. What is the minimum possible value of R?

   Proposed by James Yang

   Answer. 2240

   Solution. Basically consider “lines” of parallelograms that connect opposite parallel sides of the 8960-gon. Rectangles are forced when 2 lines corresponding to perpendicular pairs of sides intersect. This occurs 2240 distinct times. A construction for 2240 rectangles exists, just consider ”contracting” the polygon wrt some parallel pair of sides, and tiling the difference. Then just do this like 4480 times.

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9. Let $p(k)$ be the probability that if we choose a uniformly random subset S of $\{1, 2, \ldots, 18\}$, then $|S| \equiv k \pmod 5$.

Evaluate

$$\sum_{k=0}^{4} \left| p(k) - \frac{1}{5} \right| .$$

Proposed by Ishin Shah

Answer. $\dfrac{9349}{327680}$

Solution. Let p_n be the probability for our set being $\{1, 2 \cdots, n\}$.

First, note that $p_k(3k-l) = p_k(3k+l)$ for any given l as

Note that for any l, we know $p(3k-l)$ is the sum of binomials in the form $\binom{k}{i}$, where $i \equiv 3k-l \mod 5$. This is the same sum as binomials in the form $\binom{k}{k-i}$. Note that $k - i = -2k - l \equiv 3k - l$, so $p(3k-l) = p(3k+l)$.

For a given k, let $a_k = p_k(3k), b_k = p_k(3k+1), c_k = p_k(3k+2)$.

We have the following recurrences:

$$a_k = p_k(3k) = \frac{p_{k-1}(3k) + p_{k-1}(3k-1)}{2} = \frac{p_{k-1}(3(k-1)+3) + p_{k-1}(3(k-1)+2)}{2} = c_{k-1}$$
$$b_k = p_k(3k+1) = \frac{p_{k-1}(3k) + p_{k-1}(3k+1)}{2} = \frac{p_{k-1}(3(k-1)+3) + p_{k-1}(3(k-1)+4)}{2} = \frac{b_{k-1} + c_{k-1}}{2}$$
$$c_k = p_k(3k+2) = \frac{p_{k-1}(3k+2) + p_{k-1}(3k+1)}{2} = \frac{p_{k-1}(3(k-1)+5) + p_{k-1}(3(k-1)+4)}{2} = \frac{a_{k-1} + b_{k-1}}{2}$$

Our starting values are $(a_0, b_0, c_0) = (1, 0, 0)$. Note that if we subtract $\dfrac{1}{5}$ from each of them, the recurrences still hold.

We could also make our calculations simpler if we multiply by $5*2^k$, so we could make $a'_k = 5(a_k - \frac{1}{5}) * 2^k, b'_k = 5(b_k - \frac{1}{5}) * 2^k, c'_k = 5(c_k - \frac{1}{5}) * 2^k$.

Our new recurrences are $a'_k = 2c'_{k-1}, b'_k = b'_{k-1} + c'_{k-1}, c'_k = a'_{k-1} + b'_{k-1}$

This gets $c'_k = b'_{k-1} + 2c'_{k-2}$ so $b_{k-1} = c'_k - 2c'_{k-2}$. Combining this with both equations gives

$c'_{k+1} = c'_k + 3'_c k - 1 - 2c'_{k-2}$ so $c'_k = x(-\phi)^k + y(\dfrac{1}{\phi})^k + z2^k$. However, $c'_k/2^k$ goes to 0 as the probability tends to $1/5$ so $z = 0$ and we could just make this $c'_k = -c'_{k-1} + c'_{k-2}$.

This gets $c'_k = 2c'_{k-2} - c'_{k-3}$ which gets $b'_{k-1} = -c'_{k-3}$ so $b'_k = -c'_{k-2}$.

Our final solution is the expression

$$\frac{(|a'_k| + 2|b'_k| + 2|c'_k|)}{2^k * 5} = \frac{(|c'_{k-2}| + |c'_{k-1}| + |c'_k|)}{2^{k-1} * 5}$$

Note that by the linear recurrence, c'_{k-2}, c'_{k-1} have different signs, so $|c'_{k-2}| + |c'_{k-1}| = |c'_{k-2} - c'_{k-1}| = |c_k|$.

Thus, our solution is $\dfrac{|c'_k|}{2^{k-2} * 5}$

We get $c'_0 = -1, c'_1 = 3$. Thus, $|c'_k|$ turns into the Lucas series.

Computing enough values of the Lucas series gets $\boxed{\dfrac{9349}{327680}}$.

10. Let a_n be the number of ways to express n as an ordered sum of powers of 3. For example, $a_4 = 3$, since

$$4 = 1+1+1+1 = 1+3 = 3+1.$$

Let b_n denote the remainder upon dividing a_n by 3. Evaluate

$$\sum_{n=1}^{3^{2025}} b_n.$$

Proposed by Alan Abraham

Answer. 4102652

Solution. One can show that $b_n = 1$ iff $n+1$ is of the form 3^k or $2 \cdot 3^k$ (for nonnegative integer k) and $b_n = 2$ iff $n+1$ is of the form $3^a + 3^b$ (for distinct nonnegative integers a, b), and thus our sum is equal to

$$\begin{aligned}\sum_{n=2}^{3^{2025}+1} b_{n-1} &= b_{3^{2025}} + b_{3^{2025}-1} - b_0 + \sum_{n=1}^{3^{2025}-1} b_{n-1} \\ &= 2+1-1+\left(1 \cdot (2025 \cdot 2) + 2 \cdot \binom{2025}{2}\right) \\ &= 2 + 2025 \cdot 2026 = \boxed{4102652}.\end{aligned}$$

To prove the claim regarding the explicit formula for b_n we use generating functions. Let $P(x) = \sum_{n=0}^{\infty} x^{3^n}$. Then in $\mathbb{F}_3[[x]]$ we have

$$\sum_{n=0}^{\infty} b_n x^n = 1 + P(x) + P(x)^2 + \cdots = (1 - P(x))^{-1}.$$

Note that we also have $P(x)^3 = P(x) - x$, so $(1 - P(x))(P(x) + P(x)^2) = x$. Hence,

$$\sum_{n=0}^{\infty} b_n x^{n+1} = P(x) + P(x)^2,$$

which proves the claim.

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11. **(Tiebreaker)** I wrote a computer program that prints out rows 0 through 100 of Pascal's triangle (the last row is $1, 100, 4950, \ldots$). Due to the way my computer stores integers, some of the numbers appear to be negative. Particularly, integer n appears negative if and only if $\lfloor n/2^{31} \rfloor$ is odd. Estimate how many negative numbers are printed out.

Proposed by Justin Hsieh and Rohan Jain

Answer. 1697