Combinatorics

Instructions

1. Do not look at the test before the proctor starts the round.
2. This test consists of 10 short-answer problems to be solved in 50 minutes. The final estimation question will be used to break ties.
3. No computational aids other than pencil/pen are permitted.
4. Write your name and team name on your answer sheet.
5. Write your answers in the corresponding lines on your answer sheet.
6. Answers must be reasonably simplified.
7. If you believe that the test contains an error, submit your protest to the 2024 CMIMC discord.

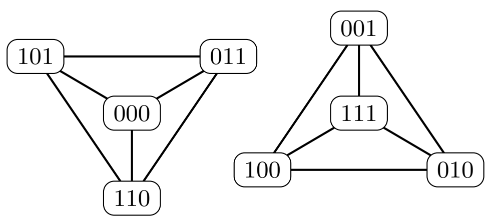
Combinatorics

1. Robert has five beads in his hand, with the letters C, M, I, M, and C, and he wants to make a circular bracelet spelling “CMIMC.” However, the power went out, so Robert can no longer see the beads in his hand. Thus, he puts the five beads on the bracelet randomly, hoping that the bracelet, when possibly rotated or flipped, spells out “CMIMC.” What is the probability that this happens? (Robert doesn’t care whether some letters appear upside down or backwards.)

2. Every day, Pinky the flamingo eats either 1 or 2 shrimp, each with equal probability. Once Pinky has consumed 10 or more shrimp in total, its skin will turn pink. Once Pinky has consumed 11 or more shrimp in total, it will get sick. What is the probability that Pinky does not get sick on the day its skin turns pink?

3. There are 34 friends are sitting in a circle playing the following game. Every round, four of them are chosen at random, and have a rap battle. The winner of the rap battle stays in the circle and the other three leave. This continues until one player remains. Everyone has equal rapping ability, i.e. every person has equal probability to win a round. What is the probability that Michael and James end up battling in the same round?

4. Let n and k be positive integers, with $k \leq n$. Define a (simple, undirected) graph $G_{n,k}$ as follows: its vertices are all of the binary strings of length n, and there is an edge between two strings if and only if they differ in exactly k positions. If $c_{n,k}$ denotes the number of connected components of $G_{n,k}$, compute
$$\sum_{n=1}^{10} \sum_{k=1}^{n} c_{n,k}.$$
(For example, $G_{3,2}$ has two connected components, as shown below.)



5. Consider a 12-card deck containing all four suits of 2, 3, and 4. A *double* is defined as two cards directly next to each other in the deck, with the same value. Suppose we scan the deck left to right, and whenever we encounter a double, we remove all the cards up to that point (including the double). Let N denote the number of times we have to remove cards. What is the expected value of N?

6. Consider a 4×4 grid of squares. We place coins in some of the grid squares so that no two coins are orthogonally adjacent, and each 2×2 square in the grid has at least one coin. How many ways are there to place the coins?

7. Alan is bored one day and decides to write down all the divisors of 1260^2 on a wall. After writing down all of them, he realizes he wrote them on the wrong wall and needs to erase all his work. Every second, he picks a random divisor which is still on the wall and instantly erases it and every number that divides it. What is the expected time it takes for Alan to erase everything on the wall?

8. Divide a regular 8960-gon into non-overlapping parallelograms. Suppose that R of these parallelograms are rectangles. What is the minimum possible value of R?

9. Let $p(k)$ be the probability that if we choose a uniformly random subset S of $\{1, 2, \ldots, 18\}$, then $|S| \equiv k \pmod 5$. Evaluate
$$\sum_{k=0}^{4} \left| p(k) - \frac{1}{5} \right|.$$

10. Let a_n be the number of ways to express n as an ordered sum of powers of 3. For example, $a_4 = 3$, since
$$4 = 1 + 1 + 1 + 1 = 1 + 3 = 3 + 1.$$
Let b_n denote the remainder upon dividing a_n by 3. Evaluate
$$\sum_{n=1}^{3^{2025}} b_n.$$

11. **(Tiebreaker)** I wrote a computer program that prints out rows 0 through 100 of Pascal’s triangle (the last row is $1, 100, 4950, \ldots$). Due to the way my computer stores integers, some of the numbers appear to be negative. Particularly, integer n appears negative if and only if $\lfloor n/2^{31} \rfloor$ is odd. Estimate how many negative numbers are printed out.