Algebra and Number Theory Round Solutions

1. Four runners are preparing to begin a 1-mile race from the same starting line. When the race starts, runners Alice, Bob, and Charlie all travel at constant speeds of 8 mph, 4 mph, and 2 mph, respectively. The fourth runner, Dave, is initially half as slow as Charlie, but Dave has a superpower where he suddenly doubles his running speed every time a runner finishes the race. How many hours does it take for Dave to finish the race?

   Proposed by Lohith Tummula

   Answer. $\frac{13}{32}$

   Solution. In the first 1/8 hours, Alice finishes the race, and Dave has only finishes an eighth of the race, since he was traveling at 1 mph. However, his speed is now doubled to 2 mph.

   In the next 1/8 hours, Bob finishes the race, and Dave travels an extra fourth of the race, with 5/8 miles to go. His speed now doubles to 4 mph, so he is now twice as fast as Charlie.

   This means that he will finish the rest of the race in

$$\frac{5}{8} \cdot \frac{1}{4} = \frac{5}{32} \ hours$$

   This means that Dave takes $1/8 + 1/8 + 5/32 = \boxed{\frac{13}{32}}$ hours.

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2. I plotted the graphs $y = (x - 0)^2$, $y = (x - 5)^2$, ... $y = (x - 45)^2$. I also draw a line $y = k$, and notice that it intersects the set of parabolas at 19 distinct points. What is k?

   Proposed by Lohith Tummala

   Answer. $\frac{2025}{4}$

   For most k values, the line intersects the set of parabolas at 20 points, two for each parabola. This special value of k for 19 intersections occur when $(x - 0)^2$ and $(x - 45)^2$ intersect:

$$(x - 0)^2 = (x - 45)^2$$

$$x = -x + 45$$

$$x = 45/2$$

   This implies that $k = y = x^2 = \boxed{\frac{2025}{4}}$.

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3. Compute $3^{3^{\cdots^{3}}} \bmod 333$, where there are 3^{3^3} 3's in the exponent.

   Proposed by Allen Yang

Answer. 36

Solution. Observe $P =$ our number is just a very large tower of 3's. We can mod bash using CRT and Fermat.

$333 = 3^2 \cdot 37$ and $P \equiv 0 \mod 9$. Then we need to find $P \mod 37$.

By Fermat, $3^{36} \equiv 1 \mod 37$ so we need to evaluate $P \mod 36$.

Since $3 \equiv -1 \mod 4$ and the exponent is odd, we have $P \equiv -1 \mod 4$ and $P \equiv 0 \mod 9$, giving us $P \equiv 27 \mod 36$. and consequently, $P \equiv 3^{27} \mod 37$. We can cube 3 three times, modding it by 37 each time. Observe $27^3 \equiv (-10)^3 \equiv -1000 + 999 \equiv -1 \mod 37$, and combining this with $P \equiv 0 \mod 9$ gives us $P \equiv \boxed{36} \mod 333$.

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4. Consider the system of equations

$$\log_x y + \log_y z + \log_z x = 8$$

$$\log_{\log_y x} z = -3$$

$$\log_z y + \log_x z = 16$$

Find z.

Proposed by Lohith Tummala

Answer. 64

Solution.

From the second equation, $(\log_y x)^{-3} = z$, implying that $\log_y x = z^{-1/3}$. This means that $\log_x y = z^{1/3}$. Substitute this into the first equation to get

$$z^{1/3} + \log_y z + \log_z x = 8$$

Now, let's rewrite the third equation:

$$\log_z y + \log_x z = \frac{1}{\log_y z} + \frac{1}{\log_z x} = \frac{\log_y z + \log_z x}{\log_y z \cdot \log_z x} = 16$$

$$\implies \log_y z + \log_z x = 16 \cdot (\log_y z \cdot \log_z x) = 16 \cdot \log_y x$$

$$\implies \log_y z + \log_z x = 16z^{-1/3}$$

Plug this into the new first equation to get

$$z^{1/3} + 16z^{-1/3} = 8$$

If we let $a = z^{1/3}$, we get a quadratic:

$$a + \frac{16}{a} = 8 \implies a^2 + 16 - 8a = 0$$

Thus, $a = 4$, so $z = \boxed{64}$.

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5. Consider all positive multiples of 77 less than $1,000,000$. What is the sum of all the odd digits that show up?

Proposed by Ishin Shah

Answer. 194832

Solution.

Note that $77 * 12987 = 999,999$.

Then, note that if I take some multiple of 77 and multiply it by 10 and subtract $999,999$ times its billions digit, I get the same number but its digit rotated. Ex: $123,508$ and $235,081$ are both multiples.

Thus, to see the number of times a number appears in some position, we could rotate that position back to the ones digit. This means to count the number of times a digit appears, we count the number of times it appears in the ones digit and multiply by 9 (This doesn't work for counting the number of times 0 appears due to leading 0s but we don't need them anyways).

Then, each of the odd digits appear 1299 times except for 3 which appears 1298 times as a ones digits since $77 * 12989 = 1,000,153$.

Thus, we get $1299 * (1 + 3 + 5 + 7 + 9) - 3 = 32472$ to count appearances in the ones digit and multiply by 6 to get $\boxed{194832}$ to get the final amount.

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6. Real numbers x and y are chosen independently and uniformly at random from the interval $[-1, 1]$. Find the probability that

$$|x| + |y| + 1 \leq 3 \min\{|x + y + 1|, |x + y - 1|\}.$$

Proposed by Justin Hsieh

Answer. $\frac{5}{16}$

Solution.

We will find the area of the region of square $[-1, 1]^2$ satisfying the given condition.

Let $c = x + y$. We can assume that $c \geq 0$; the case $c \leq 0$ is symmetric since $\min\{|c+1|, |c-1|\} = ||c| - 1|$.

For a given value of $c \geq 0$, we want to find all $x, y \in [-1, 1]$ such that $x + y = c$ and $|x| + |y| \leq 3|c - 1| - 1$. First of all, by the triangle inequality this is only possible when $c \leq 3|c - 1| - 1$, so $c \leq \frac{1}{2}$ or $c \geq 2$.

If $c \leq \frac{1}{2}$, we have $|x| + |y| \leq 2 - 3c$, since $|c - 1| = 1 - c$. The solutions lie on a segment of length $\sqrt{2}(2 - 3c)$, contained entirely within $[-1, 1]^2$ (particularly, the line segment between $(-1 + 2c, 1 - c)$ and $(1 - c, -1 + 2c)$). Thus the region of solutions for (x, y) is a trapezoid with bases $2\sqrt{2}$ and $\frac{1}{2}\sqrt{2}$, and height $\frac{1}{2} \cdot \frac{1}{\sqrt{2}}$. This has area $\frac{5}{8}$.

If $c \geq 2$, then the only possible solution is $(x, y) = (1, 1)$. This does not contribute to the area of valid solutions.

Accounting for the symmetric case $c \le 0$, we find that the area of valid (x, y) is $\frac{5}{4}$, and the total area of the square $[-1, 1]^2$ is 4. This makes the final answer $\boxed{\dfrac{5}{16}}$.

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7. Consider a recursively defined sequence a_n with $a_1 = 1$ such that, for $n \ge 2$, a_n is formed by appending the last digit of n to the end of a_{n-1}. For a positive integer m, let $\nu_3(m)$ be the largest integer t such that $3^t \mid m$. Compute

$$\sum_{n=1}^{810} \nu_3(a_n).$$

Proposed by Dennis Chen

Answer. 930

Solution.

Note that $\nu_3(a_n) = 1$ when the last digit of n is 2, 3, 5, or 6, as the sum of the digits of a_n is then divisible by 3 but not 9. Between 1 and 810, there are $\frac{810}{10} \cdot 4 = 324$ of these numbers, which contributes 324 to our sum.

Now note that $9 \mid a_n$ when n is of the form $10m$, $10m - 1$, or $10m - 2$ for some positive integer m. Also note that $a_{10m-1} = \frac{a_{10m}}{10}$, so $\nu_3(a_{10m-1}) = \nu_3(a_{10m})$. Thus our sum becomes

$$324 + \sum_{m=1}^{81} \left(\nu_3(a_{10m}) + \nu_3(a_{10m-1}) + \nu_3(a_{10m-2})\right),$$

which simplifies to

$$324 + 2\sum_{m=1}^{81} \nu_3(a_{10m}) + \sum_{m=1}^{81} \nu_3(a_{10m-2}).$$

We first handle $\sum \nu_3(a_{10m})$. Note that

$$\begin{aligned} a_{10m} &= 1234567890 \cdot \sum_{i=0}^{m-1} 10^{10i} \\ &= 1234567890 \cdot \frac{10^{10m} - 1}{10^{10} - 1}, \end{aligned}$$

so

$$\begin{aligned} \nu_3(a_{10m}) &= \nu_3(1234567890) + \nu_3\left(\frac{10^{10m} - 1}{10^{10} - 1}\right) \\ &= 2 + \nu_3(10^{10m} - 1) - \nu_3(10^{10} - 1) \\ &= 2 + (\nu_3(10 - 1) + \nu_3(m)) - (\nu_3(10 - 1) + \nu_3(10)) \\ &= 2 + \nu_3(m). \end{aligned}$$

Thus

$$\begin{aligned}\sum_{m=1}^{81} \nu_3(a_{10m}) &= \sum_{m=1}^{81} (2 + \nu_3(m)) \\ &= 162 + \lfloor \frac{81}{3} \rfloor + \lfloor \frac{81}{9} \rfloor + \lfloor \frac{81}{27} \rfloor + \lfloor \frac{81}{81} \rfloor \\ &= 162 + 27 + 9 + 3 + 1 \\ &= 202.\end{aligned}$$

Now we handle $\sum \nu_3(a_{10m-2})$, which is substantially harder. Note that

$$\begin{aligned}a_{10m-2} &= 12345678 + 1234567890 \cdot 10^8 \cdot \sum_{i=0}^{m-2} 10^{10i} \\ &= 12345678 + 1234567890 \cdot 10^8 \cdot \frac{10^{10(m-1)} - 1}{10^{10} - 1}.\end{aligned}$$

(You can check the final equation is valid even for $m = 1$.)

Here is the crucial claim: given a positive integer k, for every range of integers m with size 3^k, there is exactly one integer m such that $3^{k+2} \mid a_{10m-2}$.

Note that $\nu_3(12345678) = \nu_3(1234567890) = 2$, so divide a_{10m-2} by 9. This translates our desired condition to

$$3^k \mid \frac{12345678}{9} + \frac{1234567890}{9} \cdot 10^8 \cdot \frac{10^{10(m-1)} - 1}{10^{10} - 1}.$$

Stating this with modular arithmetic gives us

$$\frac{10^{10(m-1)} - 1}{10^{10} - 1} \equiv -\frac{12345678}{123456789 \cdot 10^8} \pmod{3^k}.$$

Since $\nu_3(10^{10} - 1) = 2$ by Lifting the Exponent,

$$10^{10(m-1)} - 1 \equiv -\frac{12345678 \cdot (10^{10} - 1)}{123456789 \cdot 10^8} \pmod{3^{k+2}}.$$

The actual value of the right-hand side does not matter; all that is important is that, when converted into an integer, it is divisible by 9. Let it be $9C$, where C is an integer. Then we want

$$10^{10(m-1)} \equiv 9C + 1 \pmod{3^{k+2}},$$

and rewriting this gives us

$$(10^{10})^m \equiv 9(10C + 1) + 1 \pmod{3^{k+2}}.$$

Note that by Lifting the Exponent, the smallest integer x that satisfies $(10^{10})^x \equiv 1 \pmod{3^{k+2}}$ is 3^k. Also, no matter what m is, it is evident that $(10^{10})^m$ is congruent to 1 mod 9. Thus every residue with a remainder of 1 when divided by 9 is achieved exactly once, as there are 3^k such residues.

Thus,

$$\begin{aligned}\sum_{m=1}^{81} \nu_3(a_{10m-2}) &= 81 \cdot 2 + \frac{81}{3} + \frac{81}{9} + \frac{81}{27} + \frac{81}{81} \\ &= 162 + 27 + 9 + 3 + 1 \\ &= 202.\end{aligned}$$

Our final answer is then

$$324 + 2\sum_{m=1}^{81} \nu_3(a_{10m}) + \sum_{m=1}^{81} \nu_3(a_{10m-2}) = 324 + 2 \cdot 202 + 202 = \boxed{930}.$$

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8. Let $P(x) = x^4 + 20x^3 + 29x^2 - 666x + 2025$. It is known that $P(x) > 0$ for every real x.

   There is a root r for P in the first quadrant of the complex plane that can be expressed as $r = \frac{1}{2}\left(a + bi + \sqrt{c + di}\right)$, where a, b, c, d are integers. Find $a + b + c + d$.

   Proposed by Ishin Shah and Henry Zheng

   Answer. 322

   Solution. Because $P(x)$ is strictly positive, we $P(x)$ can be written as the sum of two squares of polynomials. Thus, we get it as $(ax^2 + bx + c)^2 + (mx^2 + nx + p)^2 = 0$. Thus, we get $(ax^2 + bx + c) \pm i(mx^2 + nx + p) = 0$. By the form of the solution, it must be a root of a polynomial that is rational integers, so our polynomial at the end should be $x^2 + Mx + N$ where M and N are in the form $p + qi$ with p and q as integers. Thus, we can express P as a sum of squares of integer polynomials of degree at most 2. Since the only way to express 1 as a sum of squares is $1^2 + 0^2$, we have that

   $$P(x) = (x^2 + Ax + B)^2 + (Cx + D)^2$$

   This means $B^2 + D^2 = 2025$. Note that $2025 = 3^4 \cdot 5^2$. We can quickly verify that neither of B or D can be 0, so this means that one of them is $\pm 9 \cdot 3 = \pm 27$ and the other is $\pm 9 \cdot 4 = \pm 36$ Note that since the coefficient of x^3 is $2A$ we have that $A = 10$. The coefficient of x will be $2AB + 2CD$. So $AB + CD = -333$ This means $10B + CD = -333$. Since the LHS must be odd, it must be the case that $D = \pm 27$. We can check that this means $B = -36$, $D = 27$, and $C = 1$. So we can verify that

   $$P(x) = (x^2 + 10x - 36)^2 + (x + 27)^2$$

   Hence,

   $$P(x) = (x^2 + (10 + i)x - 36 + 27i)(x^2 + (10 - i)x - 36 - 27i)$$

This has roots $\frac{1}{2}\left(-10+e_1 i+e_2\sqrt{99-20e_1 i+144+108e_1 i}\right)$ where $e_1, e_2 \in \{-1, 1\}$. This equals

$$\frac{1}{2}\left(-10+e_1 i+e_2\sqrt{243+88e_1 i}\right)$$

The positive quadrant root will be $\frac{1}{2}\left(-10+i+\sqrt{243+88i}\right)$ which gives a sum of $\boxed{322}$.

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9. Find the largest prime factor of $45^5 - 1$.

Proposed by Henry Zheng

Answer. 2851

Solution.

Let $a = 45$. We can first notice that: $a^5 - 1 = (a-1)(a^4 + a^3 + a^2 + a + 1)$

Consider the general factorization of the following:

$$a^4 + a^3 + a^2 + a + 1 = (a^2 + 3a + 1)^2 - 5a(a+1)^2$$

Notice that in this case, 45 is 5 times a square (in fact, this works for any 5 times a square), and we get that:

$$a^5 - 1 = 44((45^2 + 3(45) + 1)^2 - 5(45)(46)^2)$$
$$a^5 = 44((2161)^2 - (690)^2) = (44)(2851)(1471)$$

and we can factor by difference of squares, which gives us that we want to find the largest factor of (44)(2851)(1471) Now, just using the square root fact, we test all primes below the square root of 2851, and we see that in fact 2851 is prime, which means that our largest prime factor is indeed $\boxed{2851}$.

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10. Let a_n be a recursively defined sequence with $a_0 = 2024$ and $a_{n+1} = a_n^3 + 5a_n^2 + 10a_n + 6$ for $n \geq 0$. Determine the value of

$$\sum_{n=0}^{\infty} \frac{2^n(a_n+1)}{a_n^2 + 3a_n + 4}.$$

Proposed by Alan Abraham

Answer. $\frac{1}{2026}$

With some divined intuition, we add 2 to both sides to see that

$$a_{n+1} + 2 = (a_n + 2)(a_n^2 + 3a_n + 4)$$

Taking the reciprocal of both sides and then using partial fraction decomposition gives

$$\frac{1}{a_{n+1}+2} = \frac{1}{2(a_n+2)} - \frac{a_n+1}{2(a_n^2+3a_n+4)}$$

$$\frac{a_n+1}{a_n^2+3a_n+4} = -2\left(\frac{1}{a_{n+1}+2} - \frac{1}{2(a_n+2)}\right)$$

$$\frac{2^n(a_n+1)}{a_n^2+3a_n+4} = -\left(\frac{2^{n+1}}{a_{n+1}+2} - \frac{2^n}{a_n+2}\right)$$

So we can deduce that the sum we wanted to evaluate telescopes. In particular, it approaches

$$\frac{2^0}{a_0+2} - \frac{2^N}{a_N+2}$$

It's clear that the second fraction approaches 0 (a_n grows REALLY fast), so our answer is $\boxed{\frac{1}{2026}}$

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11. **(Tiebreaker)** For $x \in (0,1)$, the function $f(x) = \max\{|\sin(1/x)|, |\sin(2/x)|\}$ satisfies $0 \leq f(x) \leq 1$. Estimate the average value of f on $(0,1)$, writing your answer in the form $0.abcdef$.

Proposed by Robert Trosten

Answer. 0.863978