CMIMC

Algebra and Number Theory Round

Instructions

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1. Do not look at the test before the proctor starts the round.
2. This test consists of 10 short-answer problems to be solved in 50 minutes. The final estimation question will be used to break ties.
3. No computational aids other than pencil/pen are permitted.
4. Write your name and team name on your answer sheet.
5. Write your answers in the corresponding lines on your answer sheet.
6. Answers must be reasonably simplified.
7. If you believe that the test contains an error, submit your protest to the 2025 CMIMC discord.

Algebra and Number Theory

1. Four runners are preparing to begin a 1-mile race from the same starting line. When the race starts, runners Alice, Bob, and Charlie all travel at constant speeds of 8 mph, 4 mph, and 2 mph, respectively. The fourth runner, Dave, is initially half as slow as Charlie, but Dave has a superpower where he suddenly doubles his running speed every time a runner finishes the race. How many hours does it take for Dave to finish the race?

2. I plotted the graphs $y = (x-0)^2$, $y = (x-5)^2$, ... $y = (x-45)^2$. I also draw a line $y = k$, and notice that it intersects the set of parabolas at 19 distinct points. What is k?

3. Compute $3^{3^{\cdot^{\cdot^{\cdot^{3}}}}} \bmod 333$, where there are 3^{3^3} 3's in the exponent.

4. Consider the system of equations
$$\log_x y + \log_y z + \log_z x = 8$$
$$\log_{\log_y x} z = -3$$
$$\log_z y + \log_x z = 16$$
Find z.

5. Consider all positive multiples of 77 less than $1,000,000$. What is the sum of all the odd digits that show up? (Count each instance of the odd digit separately.)

6. Real numbers x and y are chosen independently and uniformly at random from the interval $[-1, 1]$. Find the probability that
$$|x| + |y| + 1 \leq 3 \min\{|x+y+1|, |x+y-1|\}.$$

7. Consider a recursively defined sequence a_n with $a_1 = 1$ such that, for $n \geq 2$, a_n is formed by appending the last digit of n to the end of a_{n-1}. For a positive integer m, let $\nu_3(m)$ be the largest integer t such that $3^t \mid m$. Compute
$$\sum_{n=1}^{810} \nu_3(a_n).$$

8. Let $P(x) = x^4 + 20x^3 + 29x^2 - 666x + 2025$. It is known that $P(x) > 0$ for every real x.
There is a root r for P in the first quadrant of the complex plane that can be expressed as $r = \frac{1}{2}\left(a + bi + \sqrt{c + di}\right)$, where a, b, c, d are integers. Find $a + b + c + d$.

9. Find the largest prime factor of $45^5 - 1$.

10. Let a_n be a recursively defined sequence with $a_0 = 2024$ and $a_{n+1} = a_n^3 + 5a_n^2 + 10a_n + 6$ for $n \geq 0$. Determine the value of
$$\sum_{n=0}^{\infty} \frac{2^n(a_n + 1)}{a_n^2 + 3a_n + 4}.$$

11. **(Tiebreaker)** For $x \in (0, 1)$, the function $f(x) = \max\{|\sin(1/x)|, |\sin(2/x)|\}$ satisfies $0 \leq f(x) \leq 1$. Estimate the average value of f on $(0, 1)$, writing your answer in the form $0.abcdef$.