CMIMC

March 30, 2024

CMIMC MathDash

DO NOT TURN TO THE NEXT PAGE UNTIL INSTRUCTED TO DO SO.

You will have 30 minutes to complete 20 problems.

1. What is the value of $(2^{-1})^{-2}$?
2. Michael the Mouse finds a block of cheese in the shape of a regular tetrahedron (a pyramid with equilateral triangles for all faces) with side length 5 inches. He cuts a small 1-inch regular tetrahedron of cheese off each corner with a sharp knife. How many faces does the resulting solid have?
3. How many positive integers less than or equal to 1000 are divisible by 2 and 3 but not by 5?
4. How many digits are in the base 10 representation of 3^{30} given $\log 3 = 0.47712$?
5. How many ways are there to rearrange the letters of the word CMIMC such that the letter I has at most 1 unique letter as a neighbor? (for example, CMICM is an invalid arrangement)
6. The continued fraction
$$1 + \frac{2}{2 + \frac{2}{2+\ldots}}$$
can be expressed as $a + \sqrt{b}$ in simplest form. Find $a + b$
7. A semicircle is inscribed in right triangle ABC with right angle B and has diameter on AB, with one end on point B. Given that $AB = 15$ and $BC = 8$, the radius of the semicircle can be expressed as $\frac{a}{b}$ in simplest form. Find $a + b$
8. Find the last digit of the number
$$\frac{2024!}{(1012!)(2^{1012})}$$
9. Find the product of all real x that satisfy the equation
$$\frac{1}{x+1} + \frac{1}{x+2} = \frac{1}{x}$$
10. In a circle with radius 8 units, points A and B are on the circle such that the length of arc AB is $\frac{1}{3}$ of the circle's circumference. Point C is the midpoint of arc AB, and point D is on the circle so that line CD is perpendicular to line AB. If E is the point where lines AB and CD intersect, calculate the area of triangle CBE, squared.
11. Coin A is flipped two times and coin B is flipped four times. The probability that the number of heads obtained from flipping the two fair coins is the same can be expressed as $\frac{m}{n}$ in simplest form. Find $m + n$.
12. Let f be a function for which $f(3x) = x^2 + x + 1$. Find the sum of all values of z for which $f(\frac{z}{3}) = 7$

13. In triangle ABC, points M, N, and P lie on sides $\overline{AC}$, $\overline{AB}$, and $\overline{BC}$, respectively, such that $AB||MP$ and $AC||NP$. If $\angle ABC = 42^\circ$, $\angle MAN = 91^\circ$, and $\angle NMA = 47^\circ$, compute $\frac{CB}{BP}$.

14. There is 1 integer in between 300 and 400 (base 10) inclusive such that its last digit is 7 when written in bases 8, 10, and 12. Find this integer, in base 10.

15. The polynomial $x^3 - kx^2 + 20x - 15$ has 3 roots, one of which is known to be 3. Compute the greatest possible sum of the other two roots.

16. Compute the greatest constant K such that for all positive real numbers a, b, c, d measuring the sides of a cyclic quadrilateral, we have

$$\left(\frac{1}{a+b+c-d} + \frac{1}{a+b-c+d} + \frac{1}{a-b+c+d} + \frac{1}{-a+b+c+d}\right)(a+b+c+d) \geq K.$$

17. A set of tiles numbered 1 through 2024 is modified repeatedly by the following operation: remove all tiles numbered with a perfect square, and renumber the remaining tiles consecutively starting with 1. How many times must the operation be performed to reduce the number of tiles in the set to one?

18. Triangle ABC is equilateral with side length $\sqrt{3}$ and circumcenter at O. Point P is in the plane such that $(AP)(BP)(CP) = 7$. If the maximum possible value of OP is x and the minimum possible value of OP is y compute $x^6 - y^6$.

19. The Fibonacci numbers are defined as the sequence F_n with $F_0 = 1$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$. How many ways can 10 be written as an ordered sum of numbers found in the Fibonacci sequence? For example, 3 can be written as $1+1+1$, $2+1$, $1+2$, and 3, for a total of 4 ways.

20. Positive integers (not necessarily unique) are written, one on each face, on two cubes such that when the two cubes are rolled, each integer $2 \leq k \leq 12$ appears as the sum of the upper faces with probability $\frac{6-|7-k|}{36}$. Compute the greatest possible sum of all the faces on one cube.