Team Round Solutions

1. Solve for x if

$$\sqrt{x+1}+\sqrt{x}=5$$

Proposed by Eric Oh

Answer. $\frac{144}{25}$

Solution. Taking the reciprocal gives us $\sqrt{x+1}-\sqrt{x}=\frac{1}{5}$. Subtracting these equations gives $2\sqrt{x}=\frac{24}{5}$ making $\sqrt{x}=\frac{12}{5}$. Squaring both sides yields $x=\boxed{\frac{144}{25}}$.

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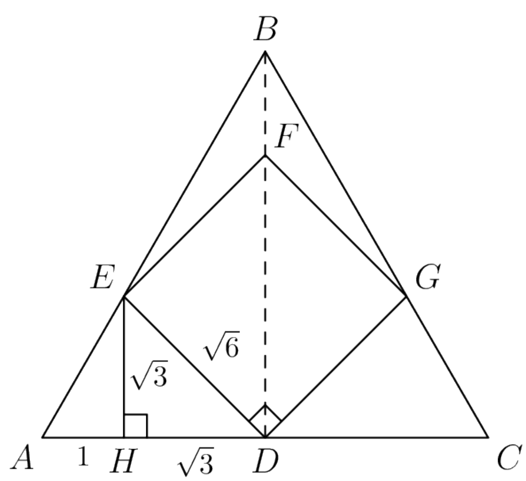
2. Let $\triangle ABC$ be equilateral. Let D be the midpoint of side AC, and let $DEFG$ be a square such that D, F, B are collinear and E, G lie on AB, CB respectively. What fraction of the area of $\triangle ABC$ is covered by square $DEFG$?

Proposed by Lohith Tummala

Answer. $2\sqrt{3}-3$

Solution.



Let H be the foot of the perpendicular from E to $\overline{AC}$. Then $\triangle AHE$ is a 30-60-90 triangle, and $\triangle DHE$ is a 45-45-90 triangle. Suppose that $AH=1$, then $EH=\sqrt{3}$, $HD=\sqrt{3}$, and $ED=\sqrt{6}$. The area of the square is 6, and the area of the triangle is

$$\frac{(2(1+\sqrt{3}))^2\cdot\sqrt{3}}{4}=6+4\sqrt{3}.$$

The desired ratio is

$$\frac{6}{6+4\sqrt{3}}=\boxed{2\sqrt{3}-3}.$$

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3. Define a function $f : \mathbb{N} \to \mathbb{N}$ to be $f(x) = (x+1)! - x!$. Find the number of positive integers $x < 49$ such that $f(x)$ divides $f(49)$.

Proposed by David Tang

Answer. 35

Solution. Each number is of the form $x(x!)$, so it works as long as x divides $49 \cdot 48 \cdot 47 \cdots (x+1)$. Obviously, we need to only check x greater than $49/2$, and the only exceptions are

$$x = 48, 47, 46, 45, 44, 43, (\text{skip } 42), 41, (\text{skip } 40), 39, 38, 37, (\text{skip } 36, 35), 34, 31, 29$$

so there are $48 - 13 = \boxed{35}$ such x.

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4. Eric and Christina are playing a game with n stones. They alternate taking some number of stones from the pile, with Eric going first. The number of stones Eric takes from the pile must be a power of 3 (e.g. 1, 3, 9, 27, ...), while the number of stones Christina takes must be a power of 2 (e.g. 1, 2, 4, 8, ...). Whoever takes the last stone wins. Find the sum of all $1 \leq n \leq 100$ for which Eric has a winning strategy.

Proposed by Connor Gordon

Answer. 121

Solution. We claim Eric only wins if he can take all the stones immediately, i.e. if the number of stones is a power of 3. This gives a sum of $1 + 3 + 9 + 27 + 81 = \boxed{121}$.

We prove that Christina can force a win if she ever gets to play, which suffices. If there are an even number of stones left when it is her turn, she can take 2 stones to keep the sum even. Since Eric has to take an odd number of stones, he cannot win on the next turn. If there are an odd number of stones left when it is her turn, she can take 1 stone to make the sum back to even. Again, Eric cannot win. Eventually they will run out of stones, and since Eric cannot win, Christina will.

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5. An ant is currently on a vertex of the top face on a 6-sided die. The ant wants to travel to the opposite vertex of the die (the vertex that is farthest from the start), and the ant can travel along edges of the die to other vertices that are on the top face of the die.

Every second, the ant picks a valid edge to move along, and the die randomly flips to an adjacent face. If the ant is on any of the bottom vertices after the flip, it is crushed and dies. What is the probability that the ant makes it to its target? (If the ant makes it to the target and the die rolls to crush it, it achieved its dreams before dying, so this counts.)

Proposed by Lohith Tummala

Answer. $\frac{1}{13}$

Solution. For $i = 1, 2, 3$, let state it be the state where the ant is currently distance i (as in shortest path along edges of the cube) from its target, which is currently on the top face of the cube. Also let state ib be the state in which the ant is distance i from the target, which is currently on the bottom face. Let P_{it} be the probability that the ant makes it to the target from state it, and define P_{ib} similarly. We are interested in P_{3b} (note that P_{3t} is not a valid state).

The key observation is that, no matter where the ant moves, it has a $\frac{1}{2}$ chance of being crushed after any move. To see this, note that there are four possible tips the die can make, exactly two of which will send the corner the ant ends up on into the ground.

With this, we can proceed with a usual states approach. From state $3b$, any step will take the ant to a distance of 2, and the ant has a $\frac{1}{2}$ of living. If the ant survives the die roll, there is a 50/50 chance whether the target will end up on the top or bottom (depending how the die rolls), so $P_{3b} = \frac{1}{4}P_{2t} + \frac{1}{4}P_{2b} + \frac{1}{2}(0)$.

From state $2t$, any step will take the ant to a distance 1 from the target. After the die rolls, there is a $\frac{1}{2}$ chance of getting crushed, and if the ant survives there is another 50/50 chance whether the target stays on top or moves to the bottom. So $P_{2t} = \frac{1}{4}P_{1t} + \frac{1}{4}P_{1b}$.

From state $2b$, there is a $\frac{1}{2}$ chance of getting crushed, a $\frac{1}{4}$ chance of making it to state $3b$, and a $\frac{1}{4}$ chance of making it to state $1t$ (after the roll), so $P_{2b} = \frac{1}{4}P_{3b} + \frac{1}{4}P_{1t}$.

From state $1t$, there is a $\frac{1}{2}$ chance of getting to the goal (even if it gets crushed after, it made it first, which counts), a $\frac{1}{4}$ chance of making it to state $2b$, and a $\frac{1}{4}$ chance of going to a distance 2 and getting crushed. So $P_{1t} = \frac{1}{2}(1) + \frac{1}{4}P_{2b}$.

From state $1b$, all moves take it to distance 2 from the target. If the ant survives, there is again a 50/50 chance whether the target will end up on the top or bottom, so $P_{1b} = \frac{1}{4}P_{2t} + \frac{1}{4}P_{2b}$.

Solving this system gives $(P_{1t}, P_{1b}, P_{2t}, P_{2b}, P_{3b}) = (\frac{7}{13}, \frac{1}{13}, \frac{2}{13}, \frac{2}{13}, \boxed{\frac{1}{13}})$.

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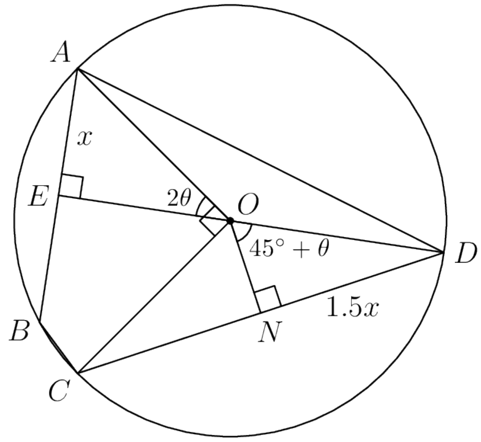
6. Cyclic quadrilateral $ABCD$ has circumradius 3. Additionally, $AC = 3\sqrt{2}$, $AB/CD = 2/3$, and $AD = BD$. Find CD.

Proposed by Justin Hsieh

Answer. $\sqrt{19} + 1$

Solution.



First, note that D and O are on the same side of $\overline{AC}$. If D were instead on the other side of $\overline{AC}$, there would be no way to make $AB/CD = 2/3$ while satisfying all of the other conditions.

Let E be the midpoint of $\overline{AB}$, and let O be the circumcenter of $ABCD$. We have that $\overline{DE}$ passes through O, since $AD = BD$. Also, $\angle COA = 90^\circ$ because $AC = 3\sqrt{2}$ and the circumradius is 3. Now let $2\theta = \angle AOE$. We have $\angle AOD = 180^\circ - 2\theta$, so

$$\angle COD = 360^\circ - \angle COA - \angle AOD = 360^\circ - 90^\circ - (180^\circ - 2\theta) = 90^\circ + 2\theta.$$

If N is the midpoint of $\overline{CD}$, we have $\angle OND = 90^\circ$ and $\angle NOD = 45^\circ + \theta$.

Let $AB = 2x$ and $CD = 3x$. We have $\sin(2\theta) = \frac{x}{3}$ from $\triangle EOA$, and $\sin(45^\circ + \theta) = \frac{1.5x}{3} = \frac{x}{2}$ from $\triangle NOD$. By angle addition on $\sin(45^\circ + \theta)$,

$$\sin(45^\circ)\cos(\theta) + \sin(\theta)\cos(45^\circ) = \frac{\sqrt{2}}{2}(\sin(\theta) + \cos(\theta)) = \frac{x}{2},$$

so $\cos(\theta) + \sin(\theta) = \frac{x}{\sqrt{2}}$. Square this to get that

$$\frac{x^2}{2} = \sin^2(\theta) + 2\sin(\theta)\cos(\theta) + \cos^2(\theta) = 1 + \sin(2\theta) = 1 + \frac{x}{3},$$

so $3x^2 - 2x - 6 = 0$. Solve for x to get $CD = 3x = \boxed{\sqrt{19} + 1}$.

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7. In the national math league, there are 7 teams. Their season is a round robin format, where each team plays other. Find the number of ways the games could go such that they have equal number of wins.

Proposed by Ishin Shah

Answer. 2640

Solution. Each team must win 3 games. Enumerate the teams $1 - 7$. There are $\binom{6}{3} = 20$ ways to put the games won by of team 1. WLOG team 1 is the winner against teams $2 - 4$ and loser against teams $5 - 7$. Now, we case upon if any of the teams $2 - 4$ beat the other $2 - 4$ teams and if if any of the teams $5 - 7$ beat the other $5 - 7$ teams.

It would look like this:

	1	2	3	4	5	6	7
1	X	L	L	L	W	W	W
2	W	X					
3	W		X				
4	W			X			
5	L				X		
6	L					X	
7	L						X

Case 1: Some team in $2 - 4$ has 2 wins in the group $2 - 4$, some team in $5 - 7$ has 2 wins in the group $5 - 7$. There are $6 \times 6 = 36$ ways to choose this. WLOG, 2 beats $3, 4$ and 3 beats 4 in the

first group. Also, WLOG 5 beats $6, 7$ and 6 beats 7. Then, 2 loses to $5, 6, 7$ and 7 beats $2, 3, 4$. Then, 4 beats $5, 6$. Then, 5 beats 3 and 3 beats 6. Thus, given the ways to choose the winners of the $2-4$ group and the $5-7$ group, this case yields 1. Thus, this case yields 36 solutions.

	1	2	3	4	5	6	7
1	X	L	L	L	W	W	W
2	W	X	W	W	L	L	L
3	W	L	X	W			L
4	W	L	L	X	W	W	L
5	L	W		L	X	W	W
6	L	W		L	L	X	W
7	L	W	W	W	L	L	X

Case 2: Some team in $2-4$ has 2 wins in the group $2-4$, each team in $5-7$ has 1 wins in the group $5-7$. There are $6 \times 2 = 12$ ways to choose this. WLOG, 2 beats $3, 4$ and 3 beats 4 in the first group. Also, WLOG 5 beats 6, 6 beats 7, and 7 beats 5 Then, 2 loses to $5, 6, 7$. From here, we choose one of 3 teams in $5, 6, 7$ to be the other win for 3 and 4 beats the other 2. This yields 3 combinations given the beginning, so this case has $3 \times 12 = 36$ solutions.

	1	2	3	4	5	6	7
1	X	L	L	L	W	W	W
2	W	X	W	W	L	L	L
3	W	L	X	W			
4	W	L	L	X			
5	L	W			X	W	L
6	L	W			L	X	W
7	L	W			W	L	X

Case 3 Some team in Every team in $2-4$ has 1 win amongst each other, and some team amongst $5-7$ has 2 wins amongst themselves. There are 12 ways for this to happen. WLOG 2 beats 3, 3 beats 4, and 4 beats 2. Also, WLOG 5 beats 6 and 7 and 6 beats 7. Then, the grid must be filled in as show, and there are 3 ways to choose who 5's last win is. After that is chosen, the rest of the grid is determined. Thus, there are 36 solutions here.

	1	2	3	4	5	6	7
1	X	L	L	L	W	W	W
2	W	X	W	L			L
3	W	L	X	W			L
4	W	W	L	X			L
5	L				X	W	W
6	L				L	X	W
7	L	W	W	W	L	L	X

Case 4 Some team in every team in $2-4$ has 1 win amongst each other, and every team in $5-7$ has 1 win amongst each other. There are 4 ways for this to happen. WLOG 2 beats 3, 3 beats 4, and 4 beats 2. Also, WLOG 5 beats 6, 6 beats 7, and 7 beats 5. Then, the grid must be filled in as show, and there are 3 ways to choose who 5's last win is. There are 6 ways to finish this as we just need some injective function from $\{2,3,4\}$ to $\{5,6,7\}$ to determine who the singular other win of the $\{2,3,4\}$ team is, for which there are 6 ways. Thus, the total in this case is 24.

	1	2	3	4	5	6	7
1	X	L	L	L	W	W	W
2	W	X	W	L			
3	W	L	X	W			
4	W	W	L	X			
5	L				X	W	L
6	L				L	X	W
7	L				W	L	X

Our final answer is

$$20(36+36+36+24) = \boxed{2640}$$

8. Compute

$$\frac{(1-\tan 10^\circ)(1-\tan 20^\circ)(1-\tan 30^\circ)(1-\tan 40^\circ)}{(1-\tan 5^\circ)(1-\tan 15^\circ)(1-\tan 25^\circ)(1-\tan 35^\circ)}.$$

Proposed by Connor Gordon

Answer. $\frac{1}{3}$

Solution. Pair the numerators and denominators up so that the angles in question add to 45°. We first compute

$$\frac{\tan(45^\circ - x) - 1}{\tan x - 1} = \frac{\frac{1-\tan x}{1+\tan x} - 1}{\tan x - 1} = \frac{2\tan x}{1 - \tan^2 x} = \tan(2x),$$

so the expression in question is equal to

$$\tan 10^\circ \tan 30^\circ \tan 50^\circ \tan 70^\circ.$$

$\tan 30^\circ$ is simply $\frac{1}{\sqrt{3}}$, so we just have to deal with the other three. In a leap of faith, we "complete the triangle" with 50° and 70° by multiplying and dividing in $\tan(60^\circ) = \sqrt{3}$. Recalling that $\tan A \tan B \tan C = \tan A + \tan B + \tan C$ for A, B, C angles of a triangle, we are interested in computing

$$\tfrac{1}{3}\tan(10^\circ)(\tan 50^\circ + \tan 60^\circ + \tan 70^\circ).$$

We can compute

$$\tan 50^\circ + \tan 60^\circ = \frac{\sin 50^\circ}{\cos 50^\circ} + \frac{\sin 50^\circ \cos 60^\circ + \sin 60^\circ \cos 50^\circ}{\cos 50^\circ \cos 60^\circ} = \frac{2\sin 110^\circ}{\cos 50^\circ},$$

and adding $\tan 70^\circ$ to this gives

$$\frac{2\sin 110^\circ \cos 70^\circ + \sin 70^\circ \cos 50^\circ}{\cos 50^\circ \cos 70^\circ}.$$

Noting that $\sin 140^\circ = \sin 40^\circ = \cos(50^\circ)$, canceling gives

$$\frac{1 + \sin 70^\circ}{\cos 70^\circ}.$$

Writing $1 = \sin(90^\circ)$, sum-to-product (and some other identities we've seen already) gives

$$\frac{2\sin 80^\circ \cos 10^\circ}{\sin 20^\circ} = \frac{2\sin 80^\circ \cos 10^\circ}{2\sin 10^\circ \cos 10^\circ} = \tan 80^\circ.$$

Going back to our original expression, this cancels out with $\tan 10^\circ = \cot(80^\circ)$, leaving us with simply $\boxed{\frac{1}{3}}$.

Alternate solution. From

$$\tan 10^\circ \tan 30^\circ \tan 50^\circ \tan 70^\circ.$$

We will now first compute

$$\sin 10^\circ \sin 50^\circ \sin 70^\circ$$

and

$$\cos 10^\circ \cos 50^\circ \cos 70^\circ$$

since $\tan 30^\circ = \frac{1}{\sqrt{3}}$.

We can get $\sin 10^\circ \sin 50^\circ \sin 70^\circ = \frac{1}{2}\sin 10^\circ(\cos 20^\circ - \cos 120^\circ) = \frac{1}{2}\sin 10^\circ(1 - 2\sin^2 10^\circ + \frac{1}{2}) = \frac{1}{4}(3\sin 10^\circ - 4\sin^3 10^\circ) = \frac{1}{4}(\sin 30^\circ) = \frac{1}{8}$ and

We can get $\cos 10^\circ \cos 50^\circ \cos 70^\circ = \frac{1}{2}\cos 10^\circ(\cos 20^\circ + \cos 120^\circ) = \frac{1}{2}\cos 10^\circ(2\cos^2 10^\circ - 1 - \frac{1}{2}) = \frac{1}{4}(4\cos^3 10^\circ + 3\cos 10^\circ) = \frac{1}{4}(\cos 30^\circ) = \frac{\sqrt{3}}{8}$

This would get

$$\tan 10^\circ \tan 50^\circ \tan 70^\circ = \frac{1}{\sqrt{3}}$$

making our final result equal to $\boxed{\frac{1}{3}}$.

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9. Suppose we have a cubic polynomial $p(x)$ such that $p(0) = 0, p(1) = 1$, and $p(x) \leq \sqrt{x}$ for $0 \leq x \leq 1$.

   Suppose $p(0.5)$ is maximized. What is the sum of $p(0.25) + p(0.75)$?

   Proposed by Ishin Shah

   Answer. $\frac{3\sqrt{2}+1}{4}$

   Solution. Express $p(x)$ as

$$p(x) = 2x\left(x - \frac{1}{2}\right) + x(x-1)(ax+b).$$

This is the most general form with $p(0) = 0$ and $p(1) = 1$, as $p(x)$ subtract the first term must be a cubic polynomial with factors x and $x - 1$.

Let us hypothesize that such a $p(x)$ exists with $p(0.5) = \sqrt{0.5}$. We can compute $p(0.5) = \sqrt{0.5}$ which implies that $a/2 + b = 2\sqrt{2}$. Then,

$$p(0.25)+p(0.75) = 2\frac{1}{4}\frac{-1}{4}+2\frac{3}{4}\frac{1}{4}+\frac{1}{4}\frac{3}{4}\left(\frac{1}{4}a+b\right)+\frac{3}{4}\frac{1}{4}\left(\frac{3}{4}a+b\right) = \frac{1}{4}+\frac{6}{16}\left(\frac{1}{2}a+b\right) = \boxed{\frac{3\sqrt{2}+1}{4}}.$$

Therefore, we did not need to use $p(x) \leq \sqrt{x}$ to obtain an answer. We only needed some mean-value property of cubics $f(x)$ that relates $f(1) + f(-1)$ with $f(-2), f(0), f(2)$.

Nevertheless, we should check that some values of a, b can actually enforce $p(x) \leq \sqrt{x}$. To do this, it is necessary for $x - p(x)^2 \geq 0$ near $x = 0.5$, which means that 0.5 must be a double root of $x - p(x)^2$. The actual expression for $p(x)$ can be computed to be

$$p(x) = \left(4 - 2\sqrt{2}\right)x^3 + \left(-4 + \sqrt{2}\right)x^2 + \left(\sqrt{2} + 1\right)x.$$

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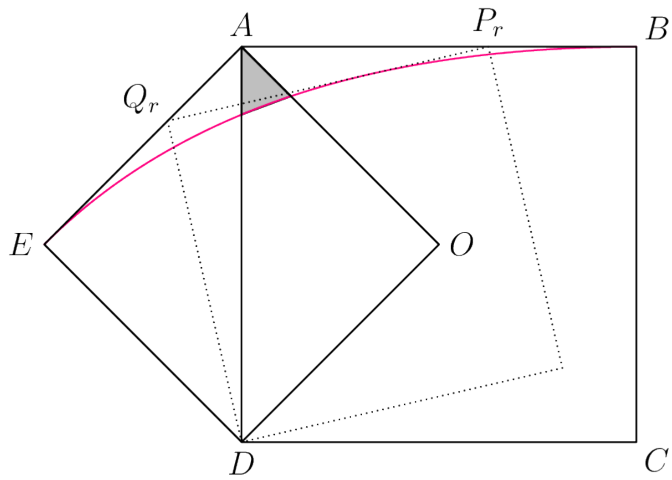
10. Square $ABCD$ has side length 2. For each $0 \leq r \leq 2$, point P_r is on side $\overline{AB}$ with $AP_r = r$, and square Σ_r is constructed with diagonal $\overline{DP_r}$. Let region $\mathcal{R}$ be the set of all points that are in both Σ_0 and Σ_2, but not in Σ_r for at least one value of r. Find the area of the convex hull of $\mathcal{R}$.

    Proposed by Justin Hsieh

    Answer. $\frac{3}{4} - \frac{\sqrt{2}}{2}$

    Solution.



Assign the following coordinates:

$$A = (0,2),\ B = (2,2),\ D = (0,0),\ P = (r,2).$$

Additionally, define point $E = (-1,1)$, which is a vertex of Σ_0.

For given P_r, let Q_r be the vertex of Σ_r which lies outside of $ABCD$. This makes $Q_0 = E$ and $Q_2 = A$. Furthermore, $\triangle EDA$ and $\triangle ADB$ are both isosceles right triangles with right angles at E and A, respectively. Then $\triangle DEQ_r \sim \triangle DAP_r$ for all r, because

$$\frac{DE}{DA} = \frac{DQ_r}{DP_r} = \frac{\sqrt{2}}{2}$$

(so $\frac{DE}{DQ_r} = \frac{DA}{DP_r}$) and

$$\angle EDQ_r = \angle EDA - (\angle Q_rDP_r - \angle ADP_r) = \angle ADP_r.$$

In particular, $\frac{EQ_r}{EA} = \frac{AP_r}{AB} = \frac{r}{2}$. This allows us to write $Q_r = (-1+\frac{r}{2}, 1+\frac{r}{2})$.

Let $O = (1,1)$ be the center of square $ABCD$. The intersection of Σ_0 and Σ_2 is $\triangle ADO$. Then $\mathcal{R}$ consists of the points in $\triangle ADO$ that are on the same side as A of parabola

$$(1-t)^2E + 2t(1-t)A + t^2B = (t^2 - 4t + 2, -t^2 + 2)\ :\ 0 \le t \le 1.$$

This parabola is the quadratic Bézier curve with control points E, A, B (see `https://en.wikipedia.org/wiki/B%C3%A9zier_curve`). Since $\frac{EQ_r}{EA} = \frac{AP_r}{AB}$ for all r, segment $\overline{P_rQ_r}$ indeed traces out this parabola.

This parabola intersects $\overline{AD}$ at $I = (0, 4\sqrt{2} - 4)$ (at $t = 2 - \sqrt{2}$), and $\overline{AO}$ at $J = (\frac{1}{4}, \frac{7}{4})$ (at $t = \frac{1}{2}$). Then the convex hull of $\mathcal{R}$ is simply $\triangle AIJ$, which has area $\boxed{\frac{3}{4} - \frac{\sqrt{2}}{2}}$ (calculated by base $AI = 6 - 4\sqrt{2}$ and (height from J to $\overline{AI}$) $= \frac{1}{4}$).

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11. **(Tiebreaker)** Submit a sequence of six digits $abcdef$, where leading zeroes are allowed. Whoever's sequence shows up *latest* in the digits of π (after the decimal point) wins the tiebreaker. For instance, 141592 would be a bad answer, as it shows up immediately. 415926 would be the second worst, as it is next.