Team Round

Instructions

1. Do not look at the test before the proctor starts the round.
2. This test consists of 10 short-answer problems to be solved in 40 minutes.
3. No computational aids other than pencil/pen are permitted.
4. Write your team name on your answer sheet.
5. Write your answers in the corresponding lines on your answer sheet.
6. Answers must be reasonably simplified.
7. If you believe that the test contains an error, submit your protest to the 2024 CMIMC discord.

Team

1. Solve for x if
$$\sqrt{x+1}+\sqrt{x}=5.$$

2. Let $\triangle ABC$ be equilateral. Let D be the midpoint of side AC, and let $DEFG$ be a square such that D, F, B are collinear and E, G lie on AB, CB respectively. What fraction of the area of $\triangle ABC$ is covered by square $DEFG$?

3. Define a function $f : \mathbb{N} \to \mathbb{N}$ to be $f(x) = (x+1)! - x!$. Find the number of positive integers $x < 49$ such that $f(x)$ divides $f(49)$.

4. Eric and Christina are playing a game with n stones. They alternate taking some number of stones from the pile, with Eric going first. The number of stones Eric takes from the pile must be a power of 3 (e.g. 1, 3, 9, 27, ...), while the number of stones Christina takes must be a power of 2 (e.g. 1, 2, 4, 8, ...). Whoever takes the last stone wins. Find the sum of all $1 \leq n \leq 100$ for which Eric has a winning strategy.

5. An ant is currently on a vertex of the top face on a 6-sided die. The ant wants to travel to the opposite vertex of the die (the vertex that is farthest from the start), and the ant can travel along edges of the die to other vertices that are on the top face of the die.

   Every second, the ant picks a valid edge to move along, and the die randomly flips to an adjacent face. If the ant is on any of the bottom vertices after the flip, it is crushed and dies. What is the probability that the ant makes it to its target? (If the ant makes it to the target and the die rolls to crush it, it achieved its dreams before dying, so this counts.)

6. Cyclic quadrilateral $ABCD$ has circumradius 3. Additionally, $AC = 3\sqrt{2}$, $AB/CD = 2/3$, and $AD = BD$. Find CD.

7. In the national math league, there are 7 teams. Their season is a round robin format, where each team plays each other team. Find the number of ways the games could go such that all teams have an equal number of wins.

8. Compute
$$\frac{(1-\tan 10^\circ)(1-\tan 20^\circ)(1-\tan 30^\circ)(1-\tan 40^\circ)}{(1-\tan 5^\circ)(1-\tan 15^\circ)(1-\tan 25^\circ)(1-\tan 35^\circ)}.$$

9. Suppose we have a cubic polynomial $p(x)$ such that $p(0) = 0, p(1) = 1$, and $p(x) \leq \sqrt{x}$ for $0 \leq x \leq 1$.

   Suppose $p(0.5)$ is maximized. What is the sum of $p(0.25) + p(0.75)$?

10. Square $ABCD$ has side length 2. For each $0 \leq r \leq 2$, point P_r is on side $\overline{AB}$ with $AP_r = r$, and square Σ_r is constructed with diagonal $\overline{DP_r}$. Let region $\mathcal{R}$ be the set of all points that are in both Σ_0 and Σ_2, but not in Σ_r for at least one value of r. Find the area of the convex hull of $\mathcal{R}$.

11. (Tiebreaker) Submit a sequence of six digits $abcdef$, where leading zeroes are allowed. Whoever's sequence shows up *latest* in the digits of π (after the decimal point) wins the tiebreaker. For instance, 141592 would be a bad answer, as it shows up immediately. 415926 would be the second worst, as it is next.