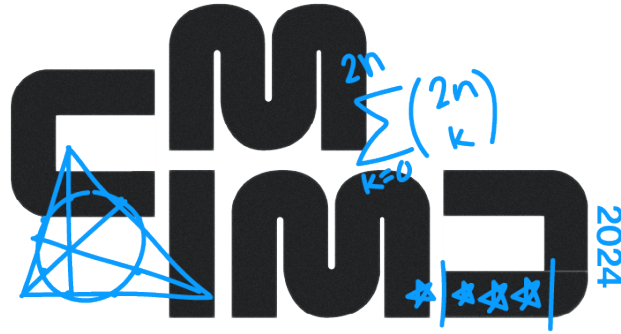




Theoretical Computer Science Round

Instructions

1. Do not look at the test before the proctor starts the round.
2. This test consists of three problems, which require proofs, to be solved within a time frame of 60 minutes. There are 300 points total.
3. No computational aids other than pencil/pen are permitted.
4. Answers should be written and clearly labeled on sheets of blank paper. Each numbered problem should be *on its own sheet*. If you have multiple pages, number them as well (e.g. 1/3, 2/3).
5. Write your team name on the upper-right corner and the problem and page number of the problem whose solution you are writing on the upper-left corner on each page you submit. Papers missing these will not be graded. Problems with more than one submission will not be graded.
6. Write legibly. Illegible handwriting will not be graded.
7. If you believe that the test contains an error, submit your protest to the 2024 CMIMC discord.





Square Finder

1. Mellon Game Lab has come up with a concept for a new game: Square Finder. The premise is as follows. You are given an $n \times n$ grid of squares (for integer $n \geq 2$), each of which is either blank or has an arrow pointing up, down, left, or right. You are also given a 2×2 grid of squares that appears somewhere in this grid, possibly rotated. For example, see if you can find the following 2×2 grid inside the larger 4×4 grid.

	↑
→	←

↑			→
	→	↑	↓
↓		↑	
↑	→		←

Did you spot it? It's in the bottom left, rotated by 90° clockwise. To make the game as interesting as possible, Mellon Game Lab would like the grid to be as large as possible and for no 2×2 grid to appear more than once in the big grid. The grid above doesn't work, as the following 2×2 grid appears twice, once in the top left corner (rotated 90° counterclockwise) and once directly below it (overlapping).

	→
↓	

Let's call a grid that avoids such repeats a *repeat-free grid*. We are interested in finding out for which n constructing an $n \times n$ repeat-free grid is possible. Here's what we know so far.

- Any 2×2 grid is repeat-free, as there is only one subgrid to worry about, and there can't possibly be any repeats.
- If we can construct an $n \times n$ repeat-free grid, we can also construct a $k \times k$ repeat-free grid for any $k \leq n$ by just taking the top left $k \times k$ of the original one we found.
- By the previous observation, if it is impossible to construct such an $n \times n$ repeat-free grid, we cannot construct a $k \times k$ repeat-free grid for any $k \geq n$, as otherwise we could take the top left $n \times n$ to get one working for n .

These three observations together tell us that either we can construct an $n \times n$ repeat-free grid for all $n \geq 2$, or there exists some upper limit $N \geq 2$ such that we can construct an $n \times n$ repeat-free grid for all $n \leq N$ but cannot construct one for any $n > N$. Your goal is to determine if such an N exists, and if so, place bounds on its value.

More precisely, this problem consists of two parts: a lower bound and an upper bound. For the lower bound, to show that $N \geq n$ for some n , you need to construct an $n \times n$ repeat-free grid (you do not need to prove your construction works). For the upper bound, to show that N is at most some value n , you must prove that it is impossible to construct an $(n + 1) \times (n + 1)$ repeat-free grid.

(Scoring details on the next page.)



Scoring

For the lower bound, a construction showing that $N \geq n$ will earn at least

- 5 points for $n = 3$
- $\lfloor 60 - 5.5(13 - n) \rfloor$ for $3 < n < 13$
- 60 points for $n = 13$

If nobody finds a construction for $n = 13$, we will scale up scores such that the largest grid obtained by any team is worth 60 points.

For the upper bound, a proof showing that $N \leq n$ will earn

- 1 point for any finite n (i.e, proving such an N exists)
- 10 points for $n = 26$
- $\lfloor 40 - 2.3(n - 13) \rfloor$ points for $13 < n < 26$
- 40 points for $n = 13$



The Decks of Cards Sponsors Sent

2. There are two different versions of this problem, each with different solutions. You must find bounds for each of these problems.
- (a) Alice and Bob are playing a collaborative game in which they agree upon an encoding/strategy. Alice shuffles a deck of 52 cards (numbered 1-52)¹ and takes the first n cards off the top of the deck. Alice then chooses one card to be put to the side, and chooses an ordering of the other $n - 1$ cards. Bob then walks into the room seeing the $n - 1$ cards in the order Alice put them in. For example, if Alice was given 9-4-10-51-7-8 and chose to put 8 to the side, she could put 5 cards in order of 9-4-10-51-7 which Bob would see. Find the lowest n for which Bob could guarantee that he could use Alice's encoding to find the card placed to the side.

Scoring

A solution resolving the problem for a given n will earn:

- 1 pt for $n = 52$
 - 10 pts for $n \leq 10$
 - 15 pts for $n = 6$
 - 25 pts for $n = 5$
- (b) Alice and Bob are playing a collaborative game in which they agree upon an encoding/strategy. Alice shuffles a deck of 52 cards (numbered 1-52)² and takes the first n cards off the top of the deck. Alice then chooses one card to be put to the side, and chooses an ordering of the other $n - 1$ cards. Carol then comes in to the room and randomly discards one of the $n - 1$ cards, and places the cards back in a way that preserves the order Alice had. Bob then walks into the room seeing the $n - 2$ cards in the order Alice put them in. For example, if Alice was given 1-2-3-4-5-6 and chose to put 6 to the side, she would put 5 cards in order of 1-2-3-4-5, and Carol removed 4, Bob would see 1-2-3-5. Find the lowest n for which Bob could guarantee that he could use Alice's encoding to find the card placed to the side.

Scoring

A solution resolving the problem for a given n will earn:

- 1 pt for $n = 52$
- 2 pts for $n < 52$
- 10 pts for $n \leq 26$
- $20 + 4(20 - n)$ pts for $10 < n \leq 20$
- 60 pts for $n \leq 10$
- 65 pts for $n \leq 9$
- 75 pts for $n \leq 8$

¹If you would prefer, you can write your solution in terms of the deck being a standard deck with 4 suits and 13 ranks, there's a way to move between them that we can handle.

²See the previous footnote.



Monty Hall (of the Arts)

3. In this problem, we explore a variant of the Monty Hall Problem.

There are n doors numbered $1, \dots, n$. A single gold coin is placed randomly behind one of the doors, with the probability it is placed behind door i equalling p_i . There are r “rounds” in which we may make at most k “turns” each, defined as follows:

During a “turn”, you pick two doors and send them to the game host. Then, the host picks one of the two doors in the following manner:

- If neither door contains the coin, the host randomly picks one with equal probability.
- If one of the doors contains the coin, the host picks the door which does not have the coin.

The host reveals that the picked door does not contain the ball, and opens it.

A “round” consists of Alice performing at least 1 and at most k turns. After all the turns in round j are complete, suppose there are d_j doors remaining. Bob will compute the probability of each individual door containing the ball. Let m_j be the minimum of these probabilities computed during the j th round. Then, Bob awards Alice $d_j m_j$ points. Note that Bob will pay Alice r times, and Alice’s total payout is the sum of the r individual payouts. Note that opened doors remain revealed between rounds.

Suppose S is some strategy that determines which doors Alice sends to the host. Let $F(S, n, k, r, (p_1, \dots, p_n))$ be the minimum possible amount Alice could earn with strategy S for parameters $(n, k, r, (p_1, \dots, p_n))$, and let

$$G(n, k, r, (p_1, \dots, p_n)) = \max_S F(S, n, k, r, (p_1, \dots, p_n)).$$

Find all tuples $(n, k, r, (p_1, \dots, p_n))$ for which $G(n, k, r, (p_1, \dots, p_n)) = r$. You may find it useful to consider lists where each element of a list is at most double the prior element.

Scoring

Subproblems:

- 3 pts Show that $G(3, 1, 1, (p_1, p_2, p_3))$ can be less than 1. (Construct some $p_1, \dots, p_3$ as well as the optimal strategy. No proof required.)
- 3 pts Show that $G(n, n - 2, 1, (p_1, \dots, p_n))$ can be less than 1 for all $n \geq 3$. (Construct some $p_1, \dots, p_n$ for each $n \geq 3$. Brief explanation of your door-picking strategy and why it is optimal suffices.)
- 3 pts: Show that $G(3, 1, 1, (p_1, \dots, p_3))$ can be equal to 1.
- 10 pts: Show that there exists a tuple $(n, k, 1, (p_1, \dots, p_n))$ with $n > 2024$ and $k > 506$ for which

$$G(n, k', 1, (p_1, \dots, p_n)) < G(n, k' + 1, 1, (p_1, \dots, p_n)) \text{ for all } k' < k - 1$$

and for which $G(n, k, 1, (p_1, \dots, p_n)) = 1$.

- 6pts: Find the tuples $(4, k, r, (p_1, \dots, p_n))$ such that $G(4, k, r, (p_1, \dots, p_n)) = r$.

For the larger problem, describe the tuples $(n, k, r, (p_1, \dots, p_n))$ for which $G(n, k, r, (p_1, \dots, p_n)) = r$ and describe the optimal strategy for one of the following constrained problems. Any unspecified variables are not constrained, except that $n \geq rk + 2$, and $p_i > 0$. (We will only grade your submission for the highest constrained problem you attempt.)

- 5 pts: Solve the problem for $k = 1$
- 10 pts: Solve the problem for $k = 2, r = 1$
- 20 pts: Solve the problem for $r = 1$ if exactly k turns must be taken during each round.
- 35 pts: Solve the problem for all $r = 1$.
- 55 pts: Solve the problem for $r = 1, 2$.
- 75 pts: Solve the problem generally (for all r .)