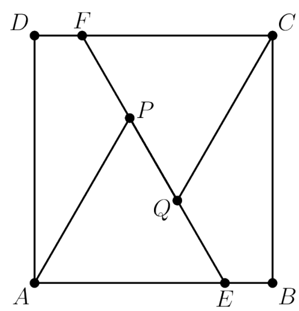
Geometry Round Solutions

1. Let $ABCD$ be a rectangle with $AB = 5$. Let E be on $\overline{AB}$ and F be on $\overline{CD}$ such that $AE = CF = 4$. Let P and Q lie inside $ABCD$ such that triangles AEP and CFQ are equilateral. If E, P, Q, and F lie on a single line, find $\overline{BC}$.

   Proposed by Connor Gordon

   Answer. $3\sqrt{3}$

   Solution.



   Consider the base and height of the rectangle as triangle CFQ slides along $\overline{EP}$. If $Q = E$, then the base and height are 6 and $2\sqrt{3}$ respectively. If $Q = P$, then the base and height are 4 and $4\sqrt{3}$ respectively. These both vary linearly, so when the base is 5, the height is $\boxed{3\sqrt{3}}$.

   Alternatively, note that the horizontal distance between P and Q is the same as the horizontal distance between E and B, which is 1. Since the horizontal distances between (F and Q) and (E and P) are both 2, the horizontal distances between (F and P) and (E and Q) are both 1, so the horizontal distance between E and F is 3. Drawing in the natural 30-60-90 triangle gives a height of $\boxed{3\sqrt{3}}$.

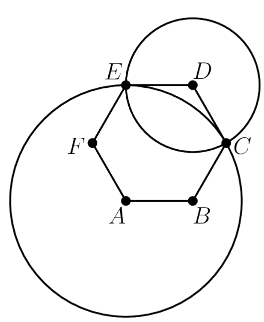
---

2. Let $ABCDEF$ be a regular hexagon of side length 1. Compute the area of the intersection of the circle centered at A passing through C and the circle centered at D passing through E.

   Proposed by Robert Trosten

   Answer. $\frac{5\pi}{6} - \sqrt{3}$

   Solution.



The first thing we note is that, since the hexagon is regular, the diagonals AC and AE have the same length. In particular, Γ_1 passes through E and Γ_2 passes through C. Consider the sectors EAC and CDE. By symmetry the triangle $\triangle EAC$ is equilateral; we can find its sidelength using the Law of Cosines to be $\sqrt{3}$. So the area of this intersection will be given by

$$\begin{gathered}(\text{area of sec. } EAC) + \text{area of sec. } CDE) \\ -(\text{area of } \triangle EAC + \text{area of } \triangle CDE).\end{gathered}$$

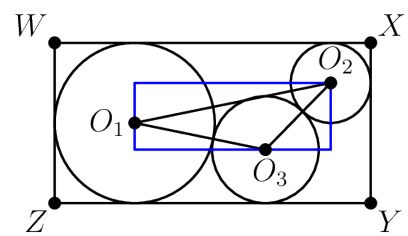
The painstaking computation (it's not so bad) yields an area of $\boxed{\frac{5\pi}{6} - \sqrt{3}}$.

---

3. Circles C_1, C_2, and C_3 are inside a rectangle $WXYZ$ such that C_1 is tangent to $\overline{WX}$, $\overline{ZW}$, and $\overline{YZ}$; C_2 is tangent to $\overline{WX}$ and $\overline{XY}$; and C_3 is tangent to $\overline{YZ}$, C_1, and C_2. If the radii of C_1, C_2, and C_3 are 1, $\frac{1}{2}$, and $\frac{2}{3}$ respectively, compute the area of the triangle formed by the centers of C_1, C_2, and C_3.

Proposed by Connor Gordon

Answer. $\frac{\sqrt{6}}{3}$



Solution. Let the centers be O_1, O_2, and O_3 respectively, and draw the bounding rectangle around $\triangle O_1O_2O_3$. The desired area is the area of this rectangle minus the area of three triangles.

For the bottom left triangle, note $O_1O_3 = \frac{5}{3}$, while the vertical leg has length $1 - \frac{2}{3} = \frac{1}{3}$. Pythagoras gives us the horizontal leg is $\frac{2\sqrt{6}}{3}$, so its area is $\frac{\sqrt{6}}{9}$.

For the bottom right triangle, note $O_2O_3 = \frac{7}{6}$, while the vertical leg has length $\frac{3}{2} - \frac{2}{3} = \frac{5}{6}$, so Pythagoras gives the horizontal leg is $\frac{\sqrt{6}}{3}$, so its area is $\frac{5\sqrt{6}}{36}$.

For the top triangle, note that the horizontal leg is $\frac{2\sqrt{6}}{3} + \frac{\sqrt{6}}{3} = \sqrt{6}$, while the vertical leg is $\frac{5}{6} - \frac{1}{3} = \frac{1}{2}$, so its area is $\frac{\sqrt{6}}{4}$.

The rectangle then has length $\sqrt{6}$ and height $\frac{5}{6}$, so its area is $\frac{5\sqrt{6}}{6}$.

Combining everything gives

$$\frac{(30-4-5-9)\sqrt{6}}{36} = \boxed{\frac{\sqrt{6}}{3}}.$$

---

4. Let ABC be an equilateral triangle with side length 1. Points D and E lie on $\overline{BC}$ and $\overline{AC}$ respectively such that $\triangle BDE$ is right isosceles, while points F and G lie on $\overline{BC}$ and $\overline{AB}$ respectively such that $\triangle CFG$ is right isosceles. Find the area of the intersection of $\triangle BDE$ and $\triangle CFG$.

Proposed by Ishin Shah

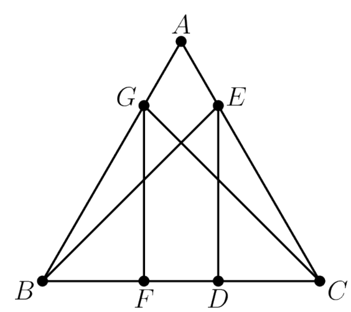
Answer. $\frac{2\sqrt{3}-3}{4}$

Solution.

Let X be the intersection of BE and CG. Since $\angle EBD = \angle FCG = 45°$, we have BXC is an isosceles right triangle with right angle at X. Since this is an iscosceles right triangle, the area of BXC is $\frac{1 \cdot 1/2}{2} = \frac{1}{4}$.

Also, let Y be the intersection between FG and BE. Then, the area of the intersection is equal to the area of BXC minus twice the area of BFY. This simplifies to $\frac{1}{4}$ minus BF^2 since twice the area of BFY is BF^2.

Now, note that $BF\sqrt{3} = GF = FC$, and since $BF + FC = 1$, we get $BF = \frac{\sqrt{3}-1}{2}$. Then, $BF^2 = \frac{4-2\sqrt{3}}{4}$. This makes our final answer $\frac{1}{4} - \frac{4-2\sqrt{3}}{4} = \boxed{\frac{2\sqrt{3}-3}{4}}$

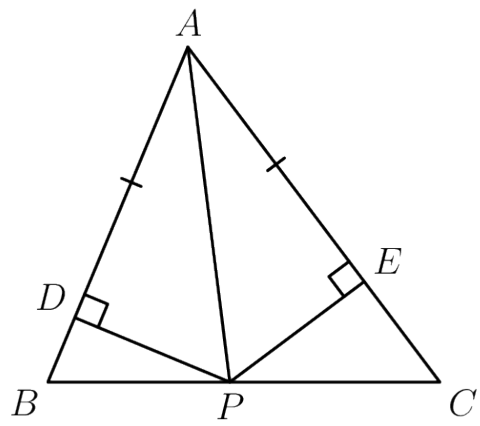


5. Triangle ABC has $AB = 13$, $BC = 14$, and $AC = 15$. Let P lie on $\overline{BC}$, and let D and E be the feet of the perpendiculars from P onto $\overline{AB}$ and $\overline{AC}$ respectively. If $AD = AE$, find this common length.

Proposed by Connor Gordon

Answer. $\frac{21}{2}$

Solution.



Note that $\triangle APD$ and $\triangle APE$ are congruent by HL, so $\overline{AP}$ is the angle bisector of $\angle A$, and $PD = PE$. By Heron's formula, the area of $\triangle ABC$ is 84. We can also express the area of $\triangle ABC$ as the sum of the areas of $\triangle APB$ and $\triangle APC$, which is

$$\frac{1}{2}AB \cdot PD + \frac{1}{2}AC \cdot PE = \frac{1}{2}(AB + AC) \cdot PD = 84.$$

This establishes $PD = PE = 6$.

By the angle bisector theorem, $PC = \frac{15}{2}$. By the Pythagorean theorem on $\triangle CEP$, we find $EC = \frac{9}{2}$, so $AE = AC - EC = \boxed{\frac{21}{2}}$.

---

6. Andrew Mellon found a piece of melon that is shaped like a octagonal prism where the bases are regular. Upon slicing it in half once, he found that he created a cross-section that is an equilateral hexagon. What is the minimum possible ratio of the height of the melon piece to the side length of the base?

Proposed by Lohith Tummala

Answer. $2\sqrt{1+\sqrt{2}}$

Solution. First, we present the optimal solution. Let $ABCDEFGH$ be the vertices of one of the octagonal bases, and let $A'B'C'D'E'F'G'H'$ be the vertices of the other octagonal bases, such that AA', BB', and so on are edges of the prism. Assuming that we fix the size of the base so that each edge of the base has side length 1, the slice that cuts through AC and $E'G'$ will have the minimum possible prism height.

The slice will cut through C and E', meaning that it will cut through the midpoint of DD' (call it X) on the way. This means that for an equilateral hexagon cross section, we want the length of AC to be equal to the length of CX. The length of AC, either through Law of Cosines or Pythagorean Theorem, can be found to be $\sqrt{2+\sqrt{2}}$. We can also use Pythagorean theorem to find CX:

$$(CD)^2 + (DX)^2 = (CX)^2$$

$$1 + \left(\frac{h}{2}\right)^2 = 2 + \sqrt{2}$$

$$h = \boxed{2\sqrt{1+\sqrt{2}}}$$

To see why this minimizes the height, notice that the only other way we can form a hexagonal cross section is to slice along A and any point on CD (call it Y), excluding C (since we just did that) and D (since that creates a rectangular cross section). Call this point Y. (This cut is mirrored across the prism center onto the other base to EZ since a cut of the prism into half must cross the center of the prism. Here, YZ is a line that goes through the center of the prism.) We see that $AY > AC$, meaning that for this hexagon to be equilateral, the two side lengths of the hexagon that traverse the non-base faces of the prism must also be longer. This necessarily increases the height. Furthermore, the length YE decreases, meaning that in the right triangle YEE', with a decrease in YE and an increase in YE' (due to the hexagon side lengths increasing), the length EE' must necessarily increase. Thus, the height increases as we move Y away from C. Thus, the minimum height is obtained with a slice through AC.

---

7. An irregular octahedron has eight faces that are equilateral triangles of side length 2. However, instead of each vertex having four "neighbors" (vertices that share an edge with it) like in a regular octahedron, for this octahedron, two of the vertices have exactly three neighbors, two of the vertices have exactly four neighbors, and two of the vertices have exactly five neighbors. Compute the volume of this octahedron.

Proposed by Connor Gordon

Answer. $2\sqrt{2}$

Solution. First, consider the two vertices that have three neighbors. We claim that the "neighborhood" of each of these vertices must look like a tetrahedron (namely the vertex and its three neighbors are the vertices of a tetrahedron).

To see this, consider one such vertex, call it A, and suppose its neighbors are B_1, B_2, and B_3. Consider the two equilateral triangular faces containing edge AB_1. Since B_2 and B_3 are the only other neighbors of A, they must be AB_1B_2 and AB_1B_3. However, each of AB_2 and AB_3 must be edges of two equilateral triangular faces, so we must also have that AB_2B_3 is an equilateral triangle. Thus $AB_1B_2B_3$ forms a tetrahedron. The same reasoning applies to the other vertex, call it A' and its neighbors B_1', B_2', and B_3'.

So now we have these two tetrahedra, which give us 8 vertices in total. This is too many, as an octahedron with triangular faces must have 6 vertices (to see this, note that we can get the number of edges via $E = \frac{8\cdot 3}{2} = 12$, as counting all of the edges of all of the faces exactly double-counts the edges of the polyhedron, and use $V + F - E = 2$ to get $V = 6$). As such, we want to "glue" these tetrahedra together so some of their vertices coincide (and then connect everything else as appropriate).

We don't want to mess with A and A', as gluing them to anything will give them more neighbors. Since we want to decrease our net number of vertices by 2, suppose we glue B_1B_2 to $B_1'B_2'$, so $B_1 = B_1'$ and $B_2 = B_2'$. We are left with six vertices, namely A, A', B_1, B_2, B_3, and B_3', and we can check that A and A' have three neighbors while B_1 and B_2 have five neighbors (they're connected to everything!). As it stands, B_3 and B_3' still only have three neighbors, but if we could connect them to each other, then they would each have four neighbors while not interfering with everything we've already done!

All that remains is to connect B_3 and B_3' in a manner that makes all of the resulting faces equilateral triangles. To do this, imagine the the two tetrahedra as being connected by a "hinge." If we move B_3 and B_3' as far apart from each other as possible, it's not too hard to see that $B_3B_3' = 2\sqrt{3} > 2$ in this case ($B_1B_3'B_2B_3$ is a rhombus with diagonals 2 and $2\sqrt{3}$). We can then swing the two tetrahedra closer together until $B_3B_3' = 2$, connect them with an edge, and note that the two new faces, $B_1B_3B_3'$ and $B_2B_3B_3'$, are equilateral triangles with side length 2. Thus we have (uniquely, up to relabeling!) constructed such an octahedron.

Now it remains to compute the volume of this octahedron. The key observation is that we can decompose $AA'B_1B_2B_3B_3'$ into three disjoint regular tetrahedra, namely $AB_1B_2B_3$, $A'B_1B_2B_3'$, and $B_1B_2B_3B_3'$ (take a moment to convince yourself of this). With this, we can easily compute the volume to be $3 \cdot \frac{2^3\sqrt{2}}{12} = \boxed{2\sqrt{2}}$.

Remark: This solid is called a tritetrahedron, or boat:

(`https://mathworld.wolfram.com/Tritetrahedron.html`).

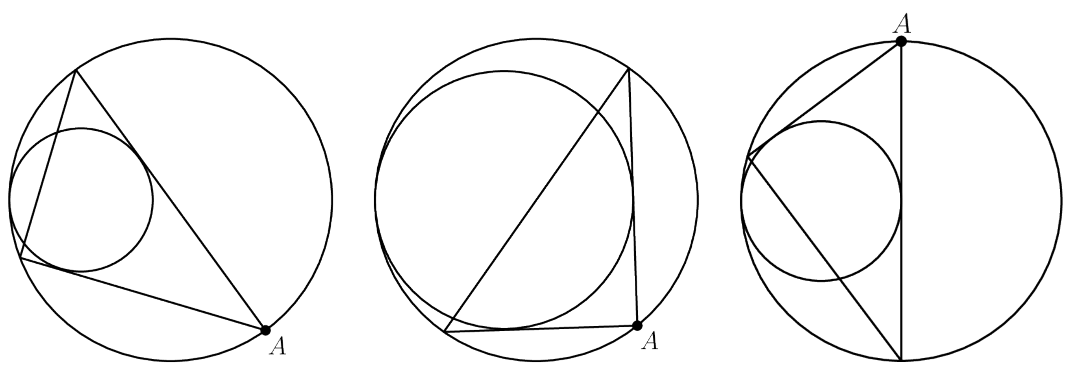
---

8. Let ω and Ω be circles of radius 1 and $R > 1$ respectively that are internally tangent at a point P. Two tangent lines to ω are drawn such that they meet Ω at only three points A, B, and C, none of which are equal to P. If triangle ABC has side lengths in a ratio of $3 : 4 : 5$, find the sum of all possible values of R.

   Proposed by Connor Gordon

   Answer. $\frac{11}{2}$

   Solution. The right angle means the hypotenuse of triangle ABC is a diameter of Ω, so the side lengths of triangle ABC are $\frac{6}{5}R$, $\frac{8}{5}R$, and $2R$. In all configurations, there will be one point, say point A, which is on both tangent lines. We can get three different configurations based on the lengths of the sides adjacent to A.



First, suppose A is the vertex between the sides of length $\frac{8}{5}R$ and $2R$, we can see that ω is the A-mixtilinear incircle of triangle ABC, which has radius $r\sec^2(\frac{A}{2})$, where r is the inradius of triangle ABC, which we can compute to be $\frac{2}{5}R$. We also compute $\sec^2(\frac{A}{2})$ to be $\frac{2}{1+\frac{4}{5}} = \frac{10}{9}$, so $1 = \frac{2}{5}R \cdot \frac{10}{9}$, which means $R = \frac{9}{4}$.

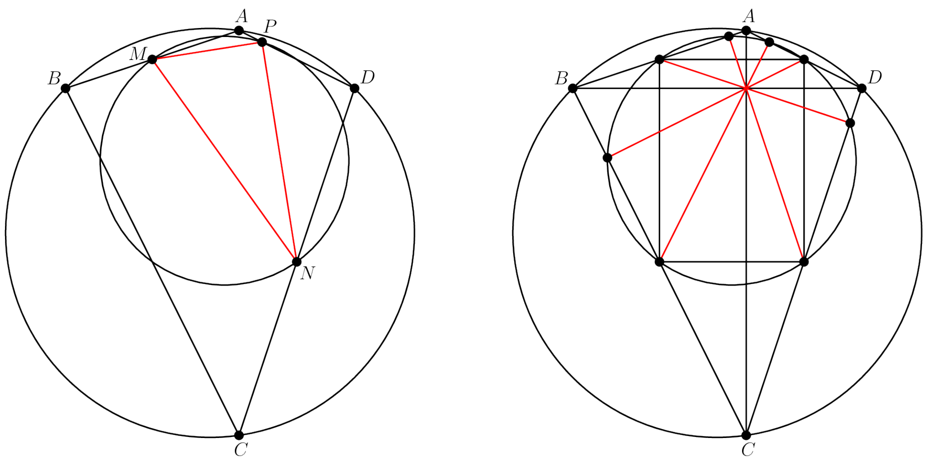
Similar calculations for when A is the right angle or between the sides of length $\frac{6}{5}R$ and $2R$ gives radii of $\frac{5}{4}$ and 2 respectively. This gives a sum of $\boxed{\frac{11}{2}}$.

9. Quadrilateral $ABCD$ is inscribed in a circle such that the midpoints of its sides also lie on a (different) circle. Let M and N be the midpoints of $\overline{AB}$ and $\overline{CD}$ respectively, and let P be the foot of the perpendicular from the intersection of $\overline{AC}$ and $\overline{BD}$ onto $\overline{BC}$. If the side lengths of $ABCD$ are 1, 3, $\sqrt{2}$, and $2\sqrt{2}$ in some order, compute the greatest possible area of the circumcircle of triangle MNP.

Proposed by Connor Gordon

Answer. $\frac{37}{40}\pi$

Solution.



Note that by similar triangles, two of the edges formed by the midpoints are parallel to $\overline{AC}$, while two of the edges formed by the midpoints are parallel to $\overline{BD}$, so the polygon formed by the midpoints is a parallelogram, and thus a rectangle since it is also cyclic. It follows that $\overline{AC}$ and $\overline{BD}$ are perpendicular.

Key Claim: The midpoints of the sides and the four perpendiculars from the intersection of the diagonals to the sides lie on a single circle.

Proof: For conveninence, label the midpoints M_1, M_2, M_3, and M_4 in order, let K be the intersection of the diagonals, and let P_1, P_2, P_3, and P_4 be the perpendiculars from M_3, M_4, M_1, and M_4 respectively onto the opposite sides (such that M_1 and P_1 lie on the same side, etc). Note that $\overline{M_1M_3}$ and $\overline{M_2M_4}$ are diameters of the circumcircle of $M_1M_2M_3M_4$ by nature of it being a rectangle.

By construction, that $\triangle M_1P_3M_3$ has a right angle at P_3, and thus P_3 lies on the circle with diameter M_1M_3, which is the circle in question. A similar argument applies to the other P_i's. $\square$

Now, we compute the circumradius of this circle. Note that the side lengths of the rectangle are half the side lengths of $\overline{AC}$ and $\overline{BC}$, so the length of the diagonals (also the diameter of the circle) is $\frac{\sqrt{AC^2+BC^2}}{2}$. Dividing by 2 gives the circumradius, and then squaring and multiplying by π gives the area, which will be $\left(\frac{AC^2+BD^2}{16}\right)\pi$.

Letting $AB = a$, $BC = b$, $CD = c$, and $DA = d$ for brevity, the formula for the lengths of the diagonals of a cyclic quadrilateral gives

$$AC^2 = \frac{(ac+bd)(ad+bc)}{(ab+cd)} \qquad BD^2 = \frac{(ac+bd)(ab+cd)}{(ad+bc)}$$

Since $ABCD$ has perpendicular diagonals $a^2 + c^2 = b^2 + d^2$, and thus the pairs of opposite sides must be $(1, 3)$ and $(\sqrt{2}, 2\sqrt{2})$. Then $ac + bd = 7$, and by symmetry it doesn't really matter how the others are assigned. As such, the "greatest possible area" was a red herring, as there is only one possible value.

We now compute it. We compute that $ab + cd$ and $ad + bc$ are $5\sqrt{2}$ and $7\sqrt{2}$ in some order, so $AC^2 + BD^2 = 7(\frac{5\sqrt{2}}{7\sqrt{2}} + \frac{7\sqrt{2}}{5\sqrt{2}}) = \frac{74}{5}$. Multiplying by $\frac{\pi}{16}$ gives $\boxed{\frac{37}{40}\pi}$.

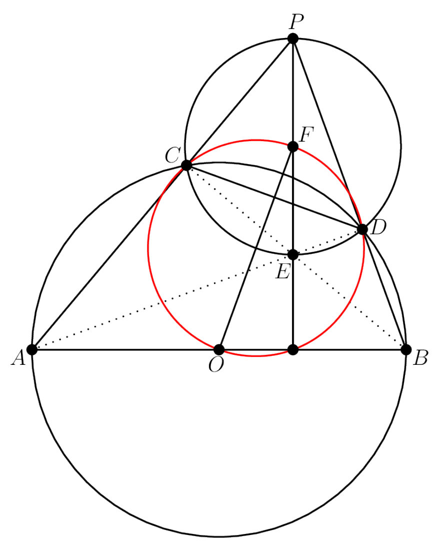
---

10. Let Ω be a unit circle with diameter AB and center O. Let C, D be on Ω and lie on the same side of AB such that $\angle CAB = 50°$ and $\angle DBA = 70°$. Suppose AD intersects BC at E. Let the perpendicular from O to CD intersect the perpendicular from E to AB at F. Find the length of OF.

Proposed by Puhua Cheng

Answer. $\frac{2\sqrt{3}}{3}$

Solution. (By Karn Chutinan)



Let AC and BD meet at P. Then we find E is the orthocenter of $\triangle PAB$. Now, remark that the circumcenter of $(PCED)$ lies on PE and the perpendicular bisector of CD, hence it is F. It thus follows that OF is a diameter of the nine-point circle of $\triangle PAB$, so OF is the circumradius of PAB.

This is not hard to compute now; law of sines gives us

$$R = \frac{AB}{2\sin\angle APB} = \frac{2}{2\sin 60°} = \frac{1}{\frac{\sqrt{3}}{2}} = \boxed{\frac{2\sqrt{3}}{3}}.$$

11. **(Tiebreaker)** A cube and a regular octahedron are inscribed in a sphere of radius 1 such that the space diagonals of the octahedron are parallel to the edges of the cube. Approximate the proportion of the sphere (by volume) that is contained in the intersection of the cube and the octahedron. Express your answer in the form $0.abcdef$.

Proposed by Connor Gordon

Answer. 0.24621355

Solution. $$\frac{\frac{4}{3}(1-3(1-\frac{1}{\sqrt{3}})^3))}{\frac{4}{3}\pi}$$