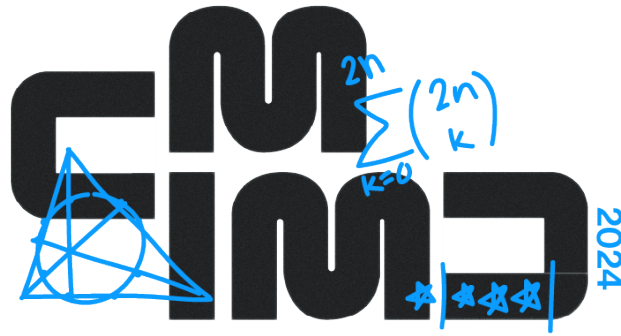




Geometry Round

Instructions

1. Do not look at the test before the proctor starts the round.
2. This test consists of 10 short-answer problems to be solved in 50 minutes. The final estimation question will be used to break ties.
3. No computational aids other than pencil/pen are permitted.
4. Write your name and team name on your answer sheet.
5. Write your answers in the corresponding lines on your answer sheet.
6. Answers must be reasonably simplified.
7. If you believe that the test contains an error, submit your protest to the 2024 CMIMC discord.





Geometry

1. Let $ABCD$ be a rectangle with $AB = 5$. Let E be on $\overline{AB}$ and F be on $\overline{CD}$ such that $AE = CF = 4$. Let P and Q lie inside $ABCD$ such that triangles AEP and CFQ are equilateral. If E , P , Q , and F lie on a single line, find $\overline{BC}$.
2. Let $ABCDEF$ be a regular hexagon of side length 1. Compute the area of the intersection of the circle centered at A passing through C and the circle centered at D passing through E .
3. Circles C_1 , C_2 , and C_3 are inside a rectangle $WXYZ$ such that C_1 is tangent to $\overline{WX}$, $\overline{ZW}$, and $\overline{YZ}$; C_2 is tangent to $\overline{WX}$ and $\overline{XY}$; and C_3 is tangent to $\overline{YZ}$, C_1 , and C_2 . If the radii of C_1 , C_2 , and C_3 are 1, $\frac{1}{2}$, and $\frac{2}{3}$ respectively, compute the area of the triangle formed by the centers of C_1 , C_2 , and C_3 .
4. Let ABC be an equilateral triangle with side length 1. Points D and E lie on $\overline{BC}$ and $\overline{AC}$ respectively such that $\triangle BDE$ is right isosceles, while points F and G lie on $\overline{BC}$ and $\overline{AB}$ respectively such that $\triangle CFG$ is right isosceles. Find the area of the intersection of $\triangle BDE$ and $\triangle CFG$.
5. Triangle ABC has $AB = 13$, $BC = 14$, and $AC = 15$. Let P lie on $\overline{BC}$, and let D and E be the feet of the perpendiculars from P onto $\overline{AB}$ and $\overline{AC}$ respectively. If $AD = AE$, find this common length.
6. Andrew Mellon found a piece of melon that is shaped like a octagonal prism where the bases are regular. Upon slicing it in half once, he found that he created a cross-section that is an equilateral hexagon. What is the minimum possible ratio of the height of the melon piece to the side length of the base?
7. An irregular octahedron has eight faces that are equilateral triangles of side length 2. However, instead of each vertex having four “neighbors” (vertices that share an edge with it) like in a regular octahedron, for this octahedron, two of the vertices have exactly three neighbors, two of the vertices have exactly four neighbors, and two of the vertices have exactly five neighbors. Compute the volume of this octahedron.
8. Let ω and Ω be circles of radius 1 and $R > 1$ respectively that are internally tangent at a point P . Two tangent lines to ω are drawn such that they meet Ω at only three points A , B , and C , none of which are equal to P . If triangle ABC has side lengths in a ratio of $3 : 4 : 5$, find the sum of all possible values of R .
9. Quadrilateral $ABCD$ is inscribed in a circle such that the midpoints of its sides also lie on a (different) circle. Let M and N be the midpoints of $\overline{AB}$ and $\overline{CD}$ respectively, and let P be the foot of the perpendicular from the midpoint of $\overline{BC}$ onto $\overline{AD}$. If the side lengths of $ABCD$ are 1, 3, $\sqrt{2}$, and $2\sqrt{2}$ in some order, compute the greatest possible area of the circumcircle of triangle MNP .
10. Let Ω be a unit circle with diameter AB and center O . Let C , D be on Ω and lie on the same side of AB such that $\angle CAB = 50^\circ$ and $\angle DBA = 70^\circ$. Suppose AD intersects BC at E . Let the perpendicular from O to CD intersect the perpendicular from E to AB at F . Find the length of OF .
11. **(Tiebreaker)** A cube and a regular octahedron are inscribed in a sphere of radius 1 such that the space diagonals of the octahedron are parallel to the edges of the cube. Approximate the proportion of the sphere (by volume) that is contained in the intersection of the cube and the octahedron. Express your answer in the form $0.abcdef$.