Combinatorics and Computer Science Round Solutions

1. For each positive integer n (written with no leading zeros), let $t(n)$ equal the number formed by reversing the digits of n. For example, $t(461) = 164$ and $t(560) = 65$. For how many three-digit positive integers m is $m + t(t(m))$ odd?

   Proposed by David Altizio

   Answer. 50

   Solution. If m has no trailing zeros, then $t(t(m)) = m$, and so $m + t(t(m)) = 2m$ is even. It follows that m is equal to $10^k m_0$ for some positive integer k. In this case, $t(m)$ results in a number with k fewer digits than m and, more specifically, $t(t(m)) = m_0$. In turn,

   $$m + t(t(m)) = 10^k m_0 + m_0 = (10^k + 1)m_0,$$

   and so m_0 must be odd.

   There are two cases to consider. If m is divisible by 100, there are five possible integers: 100, 300, 500, 700, 900. Otherwise, m is divisible by 10 but not 100, so m is of the form $\overline{p}\,\overline{q}\,0$, where q is odd. There are five choices for q and nine choices for p, yielding 45 total values for m in this case. Combining both cases yields a final answer of $\boxed{50}$.

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2. Robert has two stacks of five cards numbered 1–5, one of which is randomly shuffled while the other is in numerical order. They pick one of the stacks at random and turn over the first three cards, seeing that they are 1, 2, and 3 respectively. What is the probability the next card is a 4?

   Proposed by Connor Gordon

   Answer. $\frac{121}{122}$

   Solution. By Bayes's rule, the probability that they picked the ordered deck is

   $$\frac{\frac{1}{2}(1)}{\frac{1}{2}(1) + \frac{1}{2}(\frac{1}{60})} = \frac{60}{61},$$

   while the probability they picked the shuffled deck is $\frac{1}{61}$. The former case guarantees that 4 is the next card, while the latter case has a $\frac{1}{2}$ probability of 4 being the next card, for a total probability of $\frac{60}{61} + \frac{1}{2}(\frac{1}{61}) = \boxed{\frac{121}{122}}$.

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3. Milo rolls five fair dice which have 4, 6, 8, 12, and 20 sides respectively (and each one is labeled 1-n for appropriate n. How many distinct ways can they roll a full house (three of one number and two of another)? The same numbers appearing on different dice are considered distinct full houses, so $(1, 1, 1, 2, 2)$ and $(2, 2, 1, 1, 1)$ would both be counted.

   Proposed by Robert Trosten

Answer. 248

Solution. We will case on which dice form the three-of-a-kind (ToK). For each case, suppose the smallest die in the ToK has t sides, and the smallest die in the pair has p sides. Then the ToK can take any value 1 to t, and the pair can take any value 1 to p. Also, the ToK and pair cannot take the same value. Then the number of full houses in this case is

$$\min\{t,p\} \cdot (\max\{t,p\} - 1) = 4 \cdot (\max\{t,p\} - 1).$$

If the ToK is on the dice with...

- $4, 6, 8$ sides: $4 \cdot (12 - 1) = 44$ full houses
- $4, 6, 12$ sides: $4 \cdot (8 - 1) = 28$ full houses
- $4, 6, 20$ sides: $4 \cdot (8 - 1) = 28$ full houses
- $4, 8, 12$ sides: $4 \cdot (6 - 1) = 20$ full houses
- $4, 8, 20$ sides: $4 \cdot (6 - 1) = 20$ full houses
- $4, 12, 20$ sides: $4 \cdot (6 - 1) = 20$ full houses
- $6, 8, 12$ sides: $4 \cdot (6 - 1) = 20$ full houses
- $6, 8, 20$ sides: $4 \cdot (6 - 1) = 20$ full houses
- $6, 12, 20$ sides: $4 \cdot (6 - 1) = 20$ full houses
- $8, 12, 20$ sides: $4 \cdot (8 - 1) = 28$ full houses

Adding all of these cases, we get a total of $\boxed{248}$ full houses.

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4. There are 5 people at a party. For each pair of people, there is a $1/2$ chance they are friends, independent of all other pairs. Find the expected number of pairs of people who have a mutual friend, but are not friends themselves.

Proposed by Patrick Xue

Answer. $\frac{185}{64}$

Solution. By linearity of expectation, it suffices to find the probability that a given pair has a mutual friend but are not friends themselves, then we can multiply by the number of pairs, which is $\binom{5}{2} = 10$.

Given a pair, there is a $\frac{1}{2}$ chance they are not friends, and three potential candidates for a mutual friend. The probability that a specific other person is a mutual friend is $(\frac{1}{2})^2 = \frac{1}{4}$, so the probability that they have no mutual friends is $(\frac{3}{4})^3$. Thus, the probability they are not friends but have a mutual friend is $\frac{1}{2}(1 - (\frac{3}{4})^3) = \frac{37}{128}$. Multiplying by 10 gives $\boxed{\frac{185}{64}}$.

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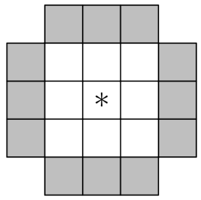
5. In the table below, place the numbers 1–12 in the shaded cells. You start at the center cell (marked with $*$). You repeatedly move up, down, left, or right, chosen uniformly at random each time, until reaching a shaded cell. Your score is the number in the shaded cell that you end up at.

Let m be the least possible expected value of your score (based on how you placed the numbers), and M be the greatest possible expected value of your score. Compute $m \cdot M$.

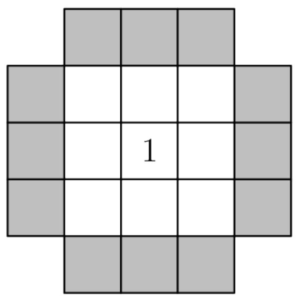


Proposed by Justin Hsieh

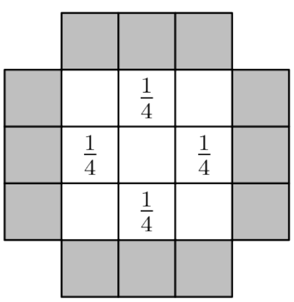
Answer. $\frac{165}{4}$

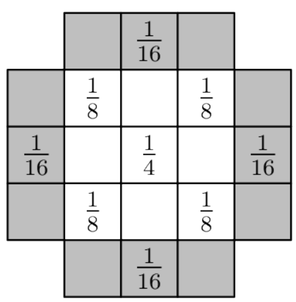
Solution. We want to find the probability of ending up at each of the shaded cells. To do this, we consider states consisting of probabilities placed in each cell. Roughly speaking, each number represents the probability of "ending up" at that cell.

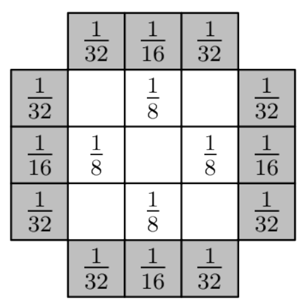
The initial state is a 1 in the center of the grid:



Given any state, we can change it by clearing any cell containing probability p, and adding $p/4$ to each orthogonally adjacent cell. This corresponds to taking one step at random. Now we can update the state so that all probabilities are in the shaded cells:







At this point, we can repeatedly apply the last two steps to see that we are twice as likely to end up at the center of a shaded 1×3 rectangle, than at any other cell. In particular, there is a $\frac{1}{8}$ chance of ending up at the center of each shaded 1×3 rectangle, and a $\frac{1}{16}$ chance of ending up at each other shaded cell:

	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{16}$	
$\frac{1}{16}$				$\frac{1}{16}$
$\frac{1}{8}$				$\frac{1}{8}$
$\frac{1}{16}$				$\frac{1}{16}$
	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{16}$	

The minimum expected score comes from placing $1, 2, 3, 4$ in the centers of the shaded rectangles, giving

$$\frac{1+2+3+4}{8} + \frac{5+6+7+8+9+10+11+12}{16} = \frac{11}{2}.$$

The maximum expected score comes from placing $9, 10, 11, 12$ in the centers of the shaded rectangles, giving

$$\frac{9+10+11+12}{8} + \frac{1+2+3+4+5+6+7+8}{16} = \frac{15}{2}.$$

The answer is $\frac{11}{2} \cdot \frac{15}{2} = \boxed{\frac{165}{4}}$.

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6. Michael and James are playing a game where they alternate throwing darts at a simplified dartboard. Each dart throw is worth either 25 points or 50 points. They track the sequence of scores per throw (which is shared between them), and on the first time the three most recent scores sum to 125, the person who threw the last dart wins. On each throw, a given player has a $2/3$ chance of getting the score they aim for, and a $1/3$ chance of getting the other score. Suppose Michael goes first, and the first two throws are both 25. If both players use an optimal strategy, what is the probability Michael wins?

Proposed by Michael Duncan

Answer. $\frac{21}{46}$

Solution. At any given point in time, only the last two scores matter. For $a, b \in \{25, 50\}$, let $p_{a,b}$ denote the probability that a player wins (playing optimally) given that it's their turn and the last two hits were a, then b. Consider first $p_{50,50}$. If this player hits a 25, they win guaranteed, and if they hits a 50, they win with probability $1 - p_{50,50}$. Of course, they'll go for the 25. So

$$p_{50,50} = \frac{2}{3} \cdot 1 + \frac{1}{3}(1 - p_{50,50})$$

and thus $p_{50,50} = \frac{3}{4}$. If the last two hits were 25 and 50, hitting a 50 would give this player a guaranteed win, so

$$p_{25,50} = \frac{2}{3} \cdot 1 + \frac{1}{3}(1 - p_{50,25}).$$

If the last two hits were 50 and 25, hitting a 50 would once again give a guaranteed win, so

$$p_{50,25} = \frac{2}{3} \cdot 1 + \frac{1}{3}(1 - p_{25,25}).$$

Now, consider the probability $p_{25,25}$. If the player aims for a 50, then

$$p_{25,25} = \frac{2}{3}(1 - p_{25,50}) + \frac{1}{3}(1 - p_{25,25}) = 1 - \frac{p_{25,25} + p_{25,50}}{3} - \frac{p_{25,50}}{3}$$

while if they aim for a 25 then

$$p_{25,25} = \frac{2}{3}(1 - p_{25,25}) + \frac{1}{3}(1 - p_{25,50}) = 1 - \frac{p_{25,25} + p_{25,50}}{3} - \frac{p_{25,25}}{3}.$$

Since the player plays optimally,

$$p_{25,25} = 1 - \frac{p_{25,25} + p_{25,50}}{3} - \frac{\min\{p_{25,25}, p_{25,50}\}}{3}.$$

The case $p_{25,50} \le p_{25,25}$ gives us a system that, when solved, yields a contradiction.

So $p_{25,50} > p_{25,25}$, and we can easily solve the resulting linear system to find that $p_{5,5} = \frac{21}{46}$.

Hence Michael wins with probability $\boxed{\frac{21}{46}}$.

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7. If $S = \{s_1, s_2, \dots, s_n\}$ is a set of integers with $s_1 < s_2 < \cdots < s_n$, define

$$f(S) = \sum_{k=1}^{n} (-1)^k k^2 s_k.$$

(If S is empty, $f(S) = 0$.) Compute the average value of $f(S)$ as S ranges over all subsets of $\{1^2, 2^2, \dots, 100^2\}$.

Proposed by Connor Gordon and Nairit Sarkar

Answer. $\frac{1}{4}$

Solution. Let $T = \{t_1, \dots, t_n\}$ be a subset of $\{2^2, 3^2, \dots, 100^2\}$. We can compute

$$f(\{1, t_1, \dots, t_n\}) = -1 + \sum_{k=1}^{n} (-1)^{k+1} (k+1)^2 t_k$$
$$f(\{t_1, \dots, t_n\}) = \sum_{k=1}^{n} (-1)^k k^2 t_k.$$

Averaging these gives $-\frac{1}{2} - \frac{1}{2}\sum_{k=1}^{n}(-1)^k(2k+1)t_k$. It then suffices to average this over all such subsets T.

Iterating this process, let $U = \{u_1, \dots, u_n\}$ be a subset of $\{3^2, 4^2, \dots, 100^2\}$. Plugging in $\{2^2, u_1, \dots, u_n\}$ and U gives

$$-\frac{1}{2} - \frac{1}{2}\left(-12 + \sum_{k=1}^{n} (-1)^{k+1}(2(k+1)+1)u_k\right)$$
$$-\frac{1}{2} - \frac{1}{2}\left(\sum_{k=1}^{n} (-1)^k (2k+1) u_k\right).$$

Averaging these gives $-\frac{1}{2}-\frac{1}{2}(-6-\sum_{k=1}^{n}(-1)^k u_k)=\frac{5}{2}+\frac{1}{2}\sum_{k=1}^{n}(-1)^k u_k$. It then suffices to average this over all such subsets U.

One last time, let $V=\{v_1,\ldots,v_n\}$ be a subset of $\{4^2,\ldots,100^2\}$. Plugging in $\{3^2,v_1,\ldots,v_n\}$ and V gives

$$\frac{5}{2}+\frac{1}{2}\left(-9+\sum_{k=1}^{n}(-1)^{k+1}u_k\right)$$
$$\frac{5}{2}+\frac{1}{2}\left(\sum_{k=1}^{n}(-1)^{k}u_k\right).$$

Averaging these gives $\frac{5}{2}+\frac{1}{2}(-\frac{9}{2})=\frac{1}{4}$, and averaging over all possible subsets V still gives $\boxed{\frac{1}{4}}$.

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8. Six assassins, numbered 1-6, stand in a circle. Each assassin is randomly assigned a target such that each assassin has a different target and no assassin is their own target. In increasing numerical order, each assassin, if they are still alive, kills their target. Find the expected number of assassins still alive at the end of this process.

Proposed by Allen Yang

Answer. $\frac{123}{53}$

Solution.

The assignment of targets is a permutation f of the numbers 1-6 such that $f(n)\neq n$ for all n (this is known as a *derangement* of six elements). It's well-known that every permutation can be decomposed into disjoint cycles, and it's clear to see that different cycles have no impact on each other with respect to assassin survival (namely, all kills/targets occur within a given cycle). We thus consider the possibilities for the "cycle types" of such permutations.

Since $f(n)\neq n$ for all n, we cannot have cycles of length 1. Thus we are interested in the ways to write 6 as a sum of positive integer greater than 1, which are $2+2+2$, $2+4$, $3+3$, and 6. Now we count the number of each type of permutation.

For three 2-cycles, there are five options for which number gets paired with 1, three options for which number gets paired with the next smallest available number, and then the last pair is fixed. This gives 15 options in total.

For two 3-cycles, there are $\binom{5}{2}=10$ options for which two numbers go with 1, and then the other trio is fixed. There are then two ways to order each of the 3-cycles, for a total of $2^2\cdot 10=40$ options.

For a 2-cycle and a 4-cycle, there are $\binom{6}{2}=15$ options for which two numbers form the pair, and then $\frac{4!}{4}=6$ ways to orient the numbers in the 4-cycle, for a total of $6\cdot 15=90$ options.

Finally, for a 6-cycle, there are $\frac{6!}{6}=120$ ways to orient the numbers in the 6-cycle.

This gives a total of $15+40+90+120=265$ derangements, which is consistent with the formula for derangements $(265=6!(\frac{1}{0!}-\frac{1}{1!}+\frac{1}{2!}-\cdots+\frac{1}{6!}))$.

Now, we compute the expected survival numbers for each cycle type. We orient the cycles such that the smallest number in the cycle is first.

For a 2-cycle (a, b), a kills b and then survives, so there is one survivor, and thus the three 2-cycle case has three survivors.

For a 3-cycle (a, b, c), a kills b before b can kill c (by the assumption that a is smallest), so c lives and then goes on to kill a, so there is also one survivor, and thus the two 3-cycle case has two survivors.

For a 4-cycle (a, b, c, d), a kills b before b can kill c, so c lives and eventually goes on to kill d. However, depending on which of c or d comes first (either occurs with probability $\frac{1}{2}$ by symmetry), d may kill a before getting killed, so we have either one or two survivors with $\frac{1}{2}$ probability each, for an expected $\frac{3}{2}$ survivors. Therefore, the expected number of survivors in the one 2-cycle and one 4-cycle case is $\frac{5}{2}$.

The 6-cycle case (a, b, c, d, e, f) requires more involved casework. We split into the following five cases, which are mutually exclusive and span all possibilities.

Case 1: $c < d$ and $e < f$ (probability $\frac{1}{4}$). In this case, b dies immediately so c lives, d dies before killing e so e lives, and f dies before killing a so a lives, leading to three survivors (a, c, and e).

Case 2: $c < d$ and $e > f$ (probability $\frac{1}{4}$). In this case, b dies immediately so c lives, d dies before killing e so e lives, and f kills a before getting killed by e, so there are two survivors (c and e).

Case 3: $c > d$ and $d < e$ (probability $\frac{1}{3}$). In this case, b dies immediately, so c lives, d kills e before c kills them or e kills f, and then f kills a, leading to two survivors (c and f).

Case 4: $c > d > e$ and $e < f$ (probability $\frac{1}{8}$). In this case, b dies immediately, so c lives, e kills f before they can kill a, then d and c kill e and d respectively, for two survivors (a and c).

Case 5: $c > d > e > f$ (probability $\frac{1}{24}$). In this case, everybody except b gets their kill off, so there is only one survivor (c).

This means that the case of a 6-cycle has an expected value of $\frac{1}{4}(3) + \frac{1}{24}(1) + \frac{17}{24}(2) = \frac{53}{24}$ survivors.

Putting all of this together, we get an expected number of

$$\frac{15(3) + 40(2) + 90(\frac{5}{2}) + 120(\frac{53}{24})}{265} = \frac{615}{265} = \boxed{\frac{123}{53}}.$$

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9. Let S denote $\{1, \ldots, 100\}$, and let f be a permutation of S such that for all $x \in S$, $f(x) \neq x$. Over all such f, find the maximum number of elements j that satisfy $\underbrace{f(\ldots(f(j))\ldots)}_{j \text{ times}} = j$.

Proposed by Hari Desikan

Answer. 80

Solution. Let an integer $z \in \{1, 2, \ldots, 100\}$ *work* iff $f^q(q) = q$. First, note that for z to work z must be in a cycle of length that divides z. Note that given any construction with a cycle of composite length c, we can replace that one cycle with cycles of length p where $p \mid c$ and still have all numbers that worked in the initial cycle work in the new cycle. We will therefore only allow for cycles of prime lengths, as any construction with cycles of composite lengths can be at least matched in number of working elements with a construction with cycles of only prime lengths.

we show that 80 numbers can work. Let $[a, b, c][d, e]$ denote that $f(a) = b, f(b) = c, f(c) = a, f(d) = e, f(e) = d$. We will separate cycles by brackets and f when evaluated at an element in a bracket yields the next element in the bracket (with wraparound.) Let p_i be the ith prime, with $p_7 = 17$ and $p_{25} = 97$.

[13,26,39,52,75,88, $p_7, p_8, p_9, p_{10}, p_{11}, p_{12}$, p_{13}] [11, 22, 33, $\ldots$, 99, p_{14}, p_{15}], [$7 \cdot 1, \ldots, 7 \cdot 7$], [$7 \cdot 8$, $7 \cdot 9$, $7 \cdot 10$, $7 \cdot 12$, $7 \cdot 13$, p_{16}] [5, 10, 15, 20, 25] [30, 40, 45, 50, 60] [70, 75, 80, 85, 90]

The idea behind this construction C is as follows: We get every composite number and every prime less than 17 to satisfy the property. We call a number n "satisfied" given a (possibly implicit) construction if $f^n(n) = n$.

Then for there to be a "better" construction, some unsatisfied number in this construction must satisfy the construction. Additionally, a better construction would see at least 81 numbers satisfied.

Suppose $p > 23$ satisfied the property. There are at most 3 multiples of p less than 100, so the rest of the numbers in the p-cycle satisfying p must be numbers that "otherwise cannot satisfy the property anyhow." Note that at least 20 $(p - 3)$ other such numbers must be used, which means we would have at least 20 numbers not satisfying the property, which means this construction can be no better than the one given.

Suppose 23 satisfies the property. Then note that from above, no larger prime can satisfy the property. Also note that 23 requires 19 unsatisfied numbers $(23 - \lfloor \frac{100}{23} \rfloor)$. There are 14 unsatisfiable numbers greater than 23 (and 1 is also unsatisfiable for 15 total,) so we must also include a number less than 23 in the construction. Then this number is not satisfied. Suppose $u \in \{19, 17, 13, 11, 1\}$ is not all in the cycle; then note that at least one more element is not satisfied (these numbers not being in the cycle guarantees that elements not in the cycle are unsatisfied.) But we only have 4 more "open spaces" in our cycle, and so one of these elements must not be in the cycle. Then any such construction cannot work.

We have thus shown that no $p \geq 23$ can be satisfied. Now, suppose 19 is satisfied. If 17 is also satisfied, then this requires too many other terms to be unsatisfied. So 17 is not satisfied, and thus every prime from 2 to 13 must be satisfied for such a construction to yield a better answer than before.

Note too that if 19 is not satisfied and 17 is satisfied, then every prime from 2 to 13 must once again be satisfied.

We now sketch the completion of the full solution at this point as the punchline becomes obvious; for 17, we need 12 "helpers", and for 13 we need at least 6 more. For 11, we need 2 more. So we need at least 20 helpers, and any case satisfying 17 can do no better than our construction. The 19 case is similar, except you need more helpers (you need at least 22 helpers total.)

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10. Suppose 100 people are gathered around at a park, each with an envelope with their name on it (all their names are distinct). Then, the envelopes are uniformly and randomly permuted between the people. If N is the number of people who end up with their original envelope, find the expected value of N^5.

Proposed by Michael Duncan

Answer. 52

Solution. Let $X_i = 1$ if the i-th person receives their own envelope, and 0 otherwise. Then $N = X_1 + X_2 + \cdots + X_{100}$.

By linearity of expectation,

$$\begin{aligned}
\mathbb{E}[N^5] &= \mathbb{E}[(X_1 + X_2 + \cdots + X_{100})^5] \\
&= \mathbb{E}[(X_1^5 + \cdots + X_{100}^5) + 5(X_1^4X_2 + \cdots + X_{100}^4X_{99}) + 10(X_1^3X_2^2 + \cdots + X_{100}^3X_{99}^2) \\
&\quad + 20(X_1^3X_2X_3 + \cdots + X_{100}^3X_{99}X_{98}) + 30(X_1^2X_2^2X_3 + \cdots + X_{100}^2X_{99}^2X_{98}) \\
&\quad + 60(X_1^2X_2X_3X_4 + \cdots + X_{100}^2X_{99}X_{98}X_{97}) + 120(X_1X_2X_3X_4X_5 + \cdots + X_{100}X_{99}X_{98}X_{97}X_{96})] \\
&= 100 \cdot \mathbb{E}[X_1^5] + 5 \cdot 100 \cdot 99 \cdot \mathbb{E}[X_1^4X_2] + 10 \cdot 100 \cdot 99 \cdot \mathbb{E}[X_1^3X_2^2] + 20 \cdot \binom{100}{3} \cdot 3 \cdot \mathbb{E}[X_1^3X_2X_3] \\
&\quad + 30 \cdot \binom{100}{3} \cdot 3 \cdot \mathbb{E}[X_1^2X_2^2X_3] + 60 \cdot \binom{100}{4} \cdot 4 \cdot \mathbb{E}[X_1^2X_2X_3X_4] + 120 \cdot \binom{100}{5} \cdot \mathbb{E}[X_1X_2X_3X_4X_5] \\
&= 100 \cdot P(X_1 = 1) + 15 \cdot 100 \cdot 99 \cdot P(X_1 = X_2 = 1) + 50 \cdot \binom{100}{3} \cdot 3 \cdot P(X_1 = X_2 = X_3 = 1) \\
&\quad + 60 \cdot \binom{100}{4} \cdot 4 \cdot P(X_1 = X_2 = X_3 = X_4 = 1) + 120 \cdot \binom{100}{5} \cdot P(X_1 = X_2 = X_3 = X_4 = X_5 = 1) \\
&= 100 \cdot \frac{99!}{100!} + 15 \cdot 100 \cdot 99 \cdot \frac{98!}{100!} + 50 \cdot \binom{100}{3} \cdot 3 \cdot \frac{97!}{100!} + 60 \cdot \binom{100}{4} \cdot 4 \cdot \frac{96!}{100!} + 120 \cdot \binom{100}{5} \cdot \frac{95!}{100!} \\
&= 1 + 15 + 25 + 10 + 1 \\
&= \boxed{52}.
\end{aligned}$$

Alternate solution. There is a somewhat motivated solution using Burnside's Lemma:

Suppose we have n people at the park instead. Consider the group action $S_n \curvearrowright [n]$, defined in the obvious way (function application). Then by Burnside's lemma, the average number of fixed points out of all permutations is simply the number of orbits of this action, which is just 1, since given any point in $[n]$, it can be mapped to any other point via some permutation. Another way to express this is, if we let $\chi(\sigma)$ denote the number of fixed points of a permutation σ under this group action, we have the number of orbits is equal to

$$\frac{1}{|S_n|} \sum_{\sigma \in S_n} \chi(\sigma) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi(\sigma)$$

However, what if we considered the diagonal group action $S_n \curvearrowright [n]^5$? This group action is defined by applying the permutation to each element in the 5 tuple. Let $\psi(\sigma)$ denote the number of fixed points under this group action of a given σ. Then, by burnside's lemma, the number of orbits of this action is equal to

$$\frac{1}{|S_n|} \sum_{\sigma \in S_n} \psi(\sigma) = \frac{1}{n!} \sum_{\sigma \in S_n} \psi(\sigma)$$

Now, what does it mean for something to be a fixed point on σ under this group action? It must be some element $(x_1, x_2, \ldots x_5) \in [n]^5$ such that $\sigma(x_i) = x_i$ for each element x_i. But we can directly count all the ways to construct such an element, it's just $\chi(\sigma)^5$, since for each coordinate we choose a fixed point of σ! Then we have that the number of orbits of this action is equal to the sum

$$\frac{1}{n!} \sum_{\sigma \in S_n} \chi(\sigma)^5$$

But this is exactly $E[N^5]$. All that is left to do is to count the orbits, which is actually the annoying part. It is not hard to see that the number of orbits is equal to the number of ways to partition 5 coordinates into groups whose values are equal, and distinct among different groups. For example, if we were working over $S_n \curvearrowright [n]^3$, the number of orbits is equal to 5, since we look at the places where we can map the elements $(1,1,1),(1,1,2),(1,2,1),(2,1,1),(1,2,3)$. Each of these elements forms a distinct orbit. (It's kind of hard to explain in words what I'm trying to say but working through this example should make it clear). This gets really annoying to count by hand but you can notice a nice recursion: Let O_k be the number of orbits of $S_n \curvearrowright [n]^k$. Then we have that

$$O_k = \sum_{i=0}^{k-1} \binom{k-1}{i} O_i$$

The motivation for this being, given a possible partition of the coordinates, removing the group containing the first coordinate will always yield a smaller set of coordinates, which are partitioned. After removing the group containing the first coordinate, the number of elements remaining can range from 0 to $k-1$, and for each of these $0 \leq i \leq k-1$ possibilities there are $\binom{k-1}{i}$ ways to position them. Then for each position, there are O_i partitions. (This recursion was based off of the wiki page for Bell Numbers).

Then we just need to solve for O_5, which is $\boxed{52}$.

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11. **(Tiebreaker)** An integer X_1 is uniformly randomly chosen from 0 to 100 inclusive. Then another integer X_2 is uniformly randomly chosen from 0 to X_1 inclusive, then X_3 from 0 to X_2 inclusive, and so on. Estimate the probability that X_5 is nonzero. Express your answer in the form $0.abcdef$.

Proposed by Connor Gordon

Answer. 0.563068

Solution. Let $p_{k,n}$ be the probability that $X_n = k$. Then

$$p_{k,n+1} = \sum_{i=k}^{100} \frac{p_{i,n}}{i+1}.$$

Starting with $p_{0,0} = \cdots = p_{99,0} = 0$ and $p_{100,0} = 1$, we compute $1 - p_{0,5} \approx 0.563068$.