Combinatorics and Computer Science Round

Instructions

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1. Do not look at the test before the proctor starts the round.
2. This test consists of 10 short-answer problems to be solved in 50 minutes. The final estimation question will be used to break ties.
3. No computational aids other than pencil/pen are permitted.
4. Write your name and team name on your answer sheet.
5. Write your answers in the corresponding lines on your answer sheet.
6. Answers must be reasonably simplified.
7. If you believe that the test contains an error, submit your protest to the 2024 CMIMC discord.

Combinatorics and Computer Science

1. For each positive integer n (written with no leading zeros), let $t(n)$ equal the number formed by reversing the digits of n. For example, $t(461) = 164$ and $t(560) = 65$. For how many three-digit positive integers m is $m + t(t(m))$ odd?

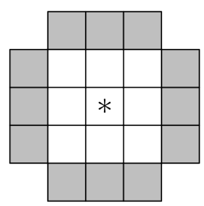
2. Robert has two stacks of five cards numbered 1–5, one of which is randomly shuffled while the other is in numerical order. They pick one of the stacks at random and turn over the first three cards, seeing that they are 1, 2, and 3 respectively. What is the probability the next card is a 4?

3. Milo rolls five fair dice which have 4, 6, 8, 12, and 20 sides respectively (and each one is labeled 1-n for appropriate n. How many distinct ways can they roll a full house (three of one number and two of another)? The same numbers appearing on different dice are considered distinct full houses, so $(1, 1, 1, 2, 2)$ and $(2, 2, 1, 1, 1)$ would both be counted.

4. There are 5 people at a party. For each pair of people, there is a $1/2$ chance they are friends, independent of all other pairs. Find the expected number of pairs of people who have a mutual friend, but are not friends themselves.

5. In the table below, place the numbers 1–12 in the shaded cells. You start at the center cell (marked with $*$). You repeatedly move up, down, left, or right, chosen uniformly at random each time, until reaching a shaded cell. Your score is the number in the shaded cell that you end up at.

   Let m be the least possible expected value of your score (based on how you placed the numbers), and M be the greatest possible expected value of your score. Compute $m \cdot M$.



6. Michael and James are playing a game where they alternate throwing darts at a simplified dartboard. Each dart throw is worth either 25 points or 50 points. They track the sequence of scores per throw (which is shared between them), and on the first time the three most recent scores sum to 125, the person who threw the last dart wins. On each throw, a given player has a $2/3$ chance of getting the score they aim for, and a $1/3$ chance of getting the other score. Suppose Michael goes first, and the first two throws are both 5. If both players use an optimal strategy, what is the probability Michael wins?

7. If $S = \{s_1, s_2, \dots, s_n\}$ is a set of integers with $s_1 < s_2 < \dots < s_n$, define
$$f(S) = \sum_{k=1}^{n} (-1)^k k^2 s_k.$$
(If S is empty, $f(S) = 0$.) Compute the average value of $f(S)$ as S ranges over all subsets of $\{1^2, 2^2, \dots, 100^2\}$.

8. Six assassins, numbered 1-6, stand in a circle. Each assassin is randomly assigned a target such that each assassin has a different target and no assassin is their own target. In increasing numerical order, each assassin, if they are still alive, kills their target. Find the expected number of assassins still alive at the end of this process.

9. Let S denote $\{1, \dots, 100\}$, and let f be a permutation of S such that for all $x \in S$, $f(x) \neq x$. Over all such f, find the maximum number of elements j that satisfy $\underbrace{f(\dots(f(j))\dots)}_{j \text{ times}} = j$.

10. Suppose 100 people are gathered around at a park, each with an envelope with their name on it (all their names are distinct). Then, the envelopes are uniformly and randomly permuted between the people. If N is the number of people who end up with their original envelope, find the expected value of N^5.

11. **(Tiebreaker)** An integer X_1 is uniformly randomly chosen from 0 to 100 inclusive. Then another integer X_2 is uniformly randomly chosen from 0 to X_1 inclusive, then X_3 from 0 to X_2 inclusive, and so on. Estimate the probability that X_5 is nonzero. Express your answer in the form $0.abcdef$.