Algebra and Number Theory Round Solutions

1. Connor is thinking of a two-digit number n, which satisfies the following properties:

   - If $n > 70$, then n is a perfect square.
   - If $n > 40$, then n is prime.
   - If $n < 80$, then the sum of the digits of n is 14.

   What is Connor's number?

   Proposed by Connor Gordon

   Answer. 59

   Solution. Note that if $n > 70$, then $n > 40$ as well, so n is both prime and a perfect square, which is nonsense. So $n \leq 70$. Thus $n < 80$, so the sum of the digits of n is 14. This means both digits of n must be at least 5, so $n > 40$ and thus n is prime. The only $40 < n \leq 70$ with digits summing to 14 are 59 and 68, of which $\boxed{59}$ is prime.

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2. Suppose $P(x) = x^2 + Ax + B$ for real A and B. If the sum of the roots of $P(2x)$ is $\frac{1}{2}$ and the product of the roots of $P(3x)$ is $\frac{1}{3}$, find $A + B$.

   Proposed by Connor Gordon

   Answer. 2

   Solution. Suppose the roots of $P(x)$ are r and s. Then the roots of $P(2x)$ are $\frac{r}{2}$ and $\frac{s}{2}$, so $\frac{r}{2} + \frac{s}{2} = \frac{1}{2} \rightarrow r + s = 1$. The roots of $P(3x)$ are $\frac{r}{3}$ and $\frac{s}{3}$, so $\frac{r}{3} \cdot \frac{s}{3} = \frac{1}{3} \rightarrow rs = 3$.

   Then we can write $P(x) = (x - r)(x - s) = x^2 - (r + s)x + rs = x^2 - x + 3$, so $A + B = \boxed{2}$.

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3. The positive integer 8833 has the property that $8833 = 88^2 + 33^2$. Find the (unique) other four-digit positive integer $\overline{abcd}$ where $\overline{abcd} = (\overline{ab})^2 + (\overline{cd})^2$.

   Proposed by Allen Yang

   Answer. 1233

   Solution. Letting $\overline{ab} = x$ and $\overline{cd} = y$, we get the equation $x^2 + y^2 = 100x + y$, or $(x - 50)^2 = -y^2 + y + 2500$. We're given that $(88, 33)$ is a solution to this equation. Note that if we change the sign of $x - 50$ while keeping y constant, we get another solution to this equation. Thus we want $x - 50 = -(88 - 50) = -38 \rightarrow x = 12$, so $\boxed{1233}$ is our answer.

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4. For positive integer n, let $f(n)$ be the largest integer k such that $k! \le n$, let $g(n) = n - (f(n))!$, and for $j \ge 1$ let

$$g^j(n) = \underbrace{g(\dots(g(n))\dots)}_{j \text{ times}}.$$

Find the smallest positive integer n such that $g^j(n) > 0$ for all $j < 30$ and $g^{30}(n) = 0$.

Proposed by Connor Gordon

Answer. 120959

Solution. This is equivalent to finding the smallest n such that its factorial base representation has digits that sum to 30. This occurs when the representation is 27654321, which corresponds to $3 \cdot 8! - 1 = \boxed{120959}$.

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5. Let

$$f(x) = (x+1)^6 + (x-1)^5 + (x+1)^4 + (x-1)^3 + (x+1)^2 + (x-1)^1 + 1.$$

Find the remainder when $\sum_{j=-126}^{126} jf(j)$ is divided by 1000.

Proposed by Hari Desikan

Answer. 626

Solution. Note that for any $j \ge 1$,

$$\begin{aligned} jf(j) + (-j)f(-j) &= j[(j+1)^6 + (j-1)^5 + (j+1)^4 + \dots + (j-1) + 1 \\ &\quad - (-j+1)^6 - (-j-1)^5 - (-j+1)^4 - \dots - (-j-1) + 1] \\ &= j[(j+1)^6 + (j-1)^5 + (j+1)^4 + \dots + (j-1) + 1 \\ &\quad - (j-1)^6 + (j+1)^5 - (j-1)^4 - \dots + (j+1) + 1]. \end{aligned}$$

Following the "zig-zags" gives two geometric series, and summing gives

$$j\left[\frac{1-(j+1)^7}{1-(j+1)} + \frac{1-(-(j-1))^7}{1-(-(j-1))}\right] = (j+1)^7 - (j-1)^7 - 2.$$

Summing this from $j = 1$ to $j = 126$, we telescope down to $127^7 + 126^7 - 1 - 252$. It then suffices to compute $127^7 + 126^7 \pmod{1000}$, as we can then subtract 253.

To do this, we use the Chinese remainder theorem. Mod 8, this expression looks like $(-1)^7 + (-2)^7 \equiv -1$, and mod 125, this expression looks like $2^7 + 1^7 \equiv 4$ mod 125. We want $-1 \equiv 7$ mod 8, and note that subtracting 125 gives us this, so our $127^7 + 126^7 \equiv -121 \equiv 879$ mod 1000. Subtracting 253 gives $\boxed{626}$.

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6. Integers a, b satisfy the following property: the line $y = 2x + ab$ passes through all intersection points of the two parabolas given by

$$y = x^2 + 2x + a, \quad y = 2x^2 + bx,$$

which intersect at least once. How many such (a, b) satisfy $|ab| \leq 100$?

Proposed by Justin Hsieh

Answer. 52

Solution. The x-coordinates of the intersections of the parabolas will satisfy

$$x^2 + 2x + a = 2x^2 + bx,$$

with solutions

$$x = \frac{2 - b \pm \sqrt{(2-b)^2 + 4a}}{2}.$$

Since the parabolas intersect at least once, we have $(2-b)^2 + 4a \geq 0$. Now considering the line $y = 2x + ab$, we have

$$y = 2x + ab = x^2 + 2x + a$$

at both possible values for x. In particular,

$$ab - a = x^2 = \frac{(2-b)^2 + (2-b)^2 + 4a \pm 2(2-b)\sqrt{(2-b)^2 + 4a}}{4}$$

for both values given by the $\pm$. Rearrange to get

$$-4a(2-b) = 2(2-b)\left((2-b) \pm \sqrt{(2-b)^2 + 4a}\right).$$

If $b = 2$, then both sides of the equation are equal to 0. This means $(a, 2)$ is a valid solution, as long as $(2-b)^2 + 4a = 4a \geq 0$. Otherwise, divide by $2(2-b)$ and rearrange:

$$-2a - 2 + b = \pm\sqrt{(2-b)^2 + 4a}.$$

Since the left side of this equation can only take one value, we must have $(2-b)^2 + 4a = 0$, which in turn makes our equation $b = 2a + 2$. Substitute $b = 2a + 2$ into $(2-b)^2 + 4a = 0$ to get $4a^2 + 4a = 0$, which makes $(a, b) = (0, 2)$ (already counted) or $(a, b) = (-1, 0)$.

Therefore the solutions are $(-1, 0)$ and $(a, 2)$ with $a \geq 0$. If $|ab| \leq 100$, then we can take a up to 50 in $(a, 2)$. This gives us a total of $\boxed{52}$ solutions.

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7. Let x_0, x_1, x_2, and x_3 be complex numbers forming a square centered at 0 in the complex plane with side length 2. For each $0 \leq k \leq 3$, there are four more complex numbers $z_{4k}, z_{4k+1}, z_{4k+2}$, and z_{4k+3} forming a square centered at x_k with side length $\sqrt{2}$. Given that $\prod_{i=0}^{15} z_i$ is a positive integer, how many possible values could it take?

Proposed by Hari Desikan

Answer. 545

Solution. Let $x_k = e^{i(\theta + \frac{k\pi}{2})}$ for $k = 0, 1, 2, 3$ and with $\theta < \frac{\pi}{2}$. Let the quadrants include the half-axes directly clockwise, and let x_i be in the $i + 1$th quadrant. Let $z_{4k+j} = x_k + s' e^{i(\alpha_k + j\frac{\pi}{2})}$ with α_k an angle in the first quadrant, for $k, j \in \{0, 1, 2, 3\}$. This means that from the origin, we first add x_k and then add a vector of the desired length and orientation to arrive at values of z

such that values of z are $\frac{\pi}{2}$ apart with respect to their respective values of x. Note additionally that $s' = \frac{1}{\sqrt{2}}$ as s' is the half diagonal of the square of side-length 1.

We will find the product of z_0, z_1, z_2, z_3 and proceed from there. This is the $k = 0$ case and has x_k in the first quadrant. Generally, we let $\omega = 4k + j$. Then,

$$\prod_{i=0}^{3} z_i = \prod_{j=0}^{3} (x_0 + s' e^{i(\alpha_0 + j\frac{\pi}{2})}) = \prod_{j=0}^{3} (x_0 - s' e^{i(\alpha_0 + j\frac{\pi}{2})}),$$

where we have noted that

$$s' e^{i(\alpha_0 + j\frac{\pi}{2})} = -s' e^{i(\alpha_0 + (j+2 \mod 4)\frac{\pi}{2})}$$

and made the change of indices $j \equiv j + 2 \mod 4$.

$$\prod_{j=0}^{3} (x_0 - s' e^{i(\alpha_0 + j\frac{\pi}{2})}) = \prod_{j=0}^{3} (x_0 - s' e^{i(\alpha_0)} e^{ij\frac{\pi}{2}}),$$

which looks remarkably like the product of the roots of unity! In fact, it is a transformation of the polynomial

$$\prod_{j=0}^{3} (x_0 - e^{ij\frac{\pi}{2}}) = x_0^4 - 1,$$

where every root is multiplied by $s' e^{i\alpha_0}$. Then

$$\prod_{j=0}^{3} (x_0 - s' e^{i(\alpha_0)} e^{ij\frac{\pi}{2}}) = \left(\left(\frac{x_0}{s' e^{i\alpha_0}} \right)^4 - 1 \right) s'^4 e^{4i\alpha_0},$$

where we multiply by the second term to keep our polynomial monic while retaining its roots. The same logic holds for $x_k : k \neq 0$ - that is,

$$\prod_{j=0}^{3} (x_k - s' e^{i(\alpha_k)} e^{ij\frac{\pi}{2}}) = \prod_{j=0}^{3} (x_0 e^{i\frac{k\pi}{2}} - s' e^{i(\alpha_k)} e^{ij\frac{\pi}{2}}) = \left(\frac{x_0 e^{i\frac{k\pi}{2}}}{s' e^{i\alpha_k}} \right)^4 - 1 = \left(\frac{x_0}{s' e^{i\alpha_k}} \right)^4 - 1,$$

as we can factor out $(e^{i\frac{k\pi}{2}})^4 = e^{2\pi k i} = 1$.

We will now re-substitute $x_0 = e^{i\theta}$ and arrive at

$$\prod_{j=0}^{3} (x_k - s' e^{i(\alpha_k)} e^{ij\frac{\pi}{2}}) = \left(\frac{e^{i\theta}}{s' e^{i\alpha_k}} \right)^4 - 1 = 4e^{4i(\theta - \alpha_k)} - 1,$$

where we have substituted $s' = \frac{1}{\sqrt{2}}$.Then

$$\prod_{\omega=0}^{15} z_\omega = \prod_{k=0}^{3} \prod_{j=0}^{3} z_\omega = \prod_{k=0}^{3} 4e^{4i(\theta - \alpha_k)} - 1.$$

We may pick θ, α_k freely in the range $[0, \frac{\pi}{2})$ and so $4(\theta - \alpha_k)$ can be any value in $[0, 2\pi)$. Let $4(\theta - \alpha_k) = \beta_k$, which can be picked freely. We now consider $\prod_{k=0}^{3} (4e^{i\beta_k} - 1)$. This is multiplying together 4 complex numbers z'_0, z'_1, z'_2, z'_3 with the sole restriction that they lie on the circle with radius 4 and center $-1+0i$ in the complex plane. It is plain to see that the minimal possible magnitude of the product of such values is 81 and the maximal possible magnitude of the product of such values is 625, and the corresponding cases $\beta_k = 0, \beta_k = \pi\ \forall k$ do result in positive integers. To construct other positive integers in the range [81, 625], simply rotate z'_0, z'_1 about -1+0i some number of radians dr and rotate z'_2, z'_3 $-dr$ radians from an initial position of $z'_1 = z'_2 = z'_3 = z'_4 = 3$. Then by symmetry the arguments of z_1, z_2 are the negatives of the arguments of z_3, z_4, so the resulting product is still real. However, the magnitude clearly sweeps out all values from 81 to 625, so all values from 81 to 625 must work - that is, $\boxed{545}$ values work.

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8. Compute the number of non-negative integers $k < 2^{20}$ such that $\binom{5k}{k}$ is odd.

Proposed by David Tang

Answer. 20736

Solution. By Lucas's Theorem, $\binom{5k}{k}$ is the same parity as $\prod_i \binom{5k_i}{k_i}$ where it's the i-th digit in binary. Thus, it is odd if and only if there is no $\binom{0}{1}$.

5 is 101_2 in binary. Visualize $5k$ as $4k+k$. If this sum has no carry, then there will be no $\binom{0}{1}$ so it works. If there is a carry, we consider the right-most carry, and that must result in a $\binom{0}{1}$. Thus, it suffices to find all binary strings with no substring of 101 or 111.

By casing on whether the first binary digits are $0, 10, 11$, we get the recursion that $f(n) = f(n-1) + f(n-3) + f(n-4)$. We also verify that the initial values are $f(0) = 1 = F_1^2, f(1) = 2 = F_1 * F_2, f(2) = 4 = F_2^2, f(3) = 6 = F_2 * F_3$. We can see that the recursion keeps this pattern. For example, we can see that $f(4) = F_2 * F_3 + F_1 * F_2 + F_1^2 = F_2 * F_3 + F_1 * F_3 = F_3^2$ and $f(5) = F_3^2 + F_2^2 + F_1 * F_2 = F_3^2 + F_3 * F_2 = F_3 * F_4$.

Thus, we have that $f(20) = F_{11}^2 = 144^2 = \boxed{20736}$.

Another way to get this is to separate odd and even bits, and we realize that no two consecutive odd or even bits can be both 1s. The number of ways to do the odd/even bits is Fibonacci, so our answer is the product of the two Fibonacci terms corresponding with the number of odd bits plus 1 and the number of even bits plus 1.

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9. Let $\mathbb{Q}_{\geq 0}$ be the non-negative rational numbers, $f : \mathbb{Q}_{\geq 0} \to \mathbb{Q}_{\geq 0}$ such that $f(z+1) = f(z) + 1$, $f(1/z) = f(z)$ for $z \neq 0$, and $f(0) = 0$. Define a sequence P_n of non-negative integers recursively via

$$P_0 = 0, \quad P_1 = 1, \quad P_n = 2P_{n-1} + P_{n-2}$$

for every $n \geq 2$. Find $f\left(\frac{P_{20}}{P_{24}}\right)$.

Proposed by Robert Trosten

Answer. 166

Solution. Solving the sequence using the characteristic polynomial, we see that there are $\alpha, \beta \in \mathbb{R}$ such that

$$P_n = \alpha(1+\sqrt{2})^n + \beta(1-\sqrt{2})^n$$

for every n. So

$$P_{3n} = \alpha(1+\sqrt{2})^{3n} + \beta(1-\sqrt{2})^{3n}.$$

Computing, $(1+\sqrt{2})^4 = 17-12\sqrt{2}$ and $(1-\sqrt{2})^4 = 17+12\sqrt{2}$. Their sum is 34 and product is 1, so we deduce the relation

$$P_{n+4} = 34P_n - P_{n-4}$$

for every $n \geq 4$. Dividing,

$$\frac{P_{n+4}}{P_n} = 34 - \frac{P_{n-4}}{P_n} = 33 + 1 - \frac{P_{n-4}}{P_n}.$$

We claim that $f(x) = f(1-x)$ for all relevant $0 < x < 1$. Indeed,

$$\begin{aligned} f(1-x) &= f\left(\frac{1}{1-x}\right) \\ &= 1 + f\left(\frac{x}{1-x}\right) \\ &= 1 + f\left(\frac{1-x}{x}\right) \\ &= f(1/x) = f(x). \end{aligned}$$

From this lemma (and easy induction) we see that

$$f\left(\frac{P_{n+4}}{P_n}\right) = 33 + f\left(1 - \frac{P_{n-4}}{P_n}\right) = 33 + f\left(\frac{P_{n-4}}{P_n}\right),$$

so that (assuming $n \neq 0$, so $f(z) = f(1/z)$) we have

$$f\left(\frac{P_{n+4}}{P_n}\right) = 33 + f\left(\frac{P_n}{P_{n-4}}\right).$$

Running this down (being careful not to trip at the end),

$$f(P_{24}/P_{20}) = 33 \cdot 4 + f(P_8/P_4) = 33 \cdot 4 + 34 - 0 = \boxed{166}$$

as desired.

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10. There exists a unique pair of polynomials $(P(x), Q(x))$ such that

$$\begin{aligned} P(Q(x)) &= P(x)(x^2 - 6x + 7) \\ Q(P(x)) &= Q(x)(x^2 - 3x - 2) \end{aligned}$$

Compute $P(10) + Q(-10)$.

Proposed by Connor Gordon

Answer. -90

Solution. Let d_P and d_Q be the degrees of P and Q respectively. The given equations imply that $d_P d_Q = d_P + 2 = d_Q + 2 \to d_P = d_Q = 2$, so P and Q are quadratic.

Now let the leading coefficients of P and Q be L_P and L_Q respectively. The given equations imply that $L_P L_Q^2 = L_P$ and $L_Q L_P^2 = L_Q$, which imply that $L_P, L_Q = \pm 1$ since $L_P, L_Q \neq 0$.

Now let the sum of the roots of P and Q be S_P and S_Q respectively. Suppose the roots of P are r and s. Then $P(Q(x)) = 0$ precisely when $Q(x) = r$ or $Q(x) = s$. These each have sum of roots S_Q (By Vieta, any polynomial $R(x)$ of degree at least 2 satisfies the property that the sum of the roots of $R(x) - c$ is the same as the sum of the roots of $R(x)$), so the sum of the roots of $P(Q(x))$ is $2S_Q$. Applying this reasoning to the given equations gives $2S_Q = S_P + 6$ and $2S_P = S_Q + 3$. Solving this system gives $S_P = 4$ and $S_Q = 5$.

So now (by some more Vieta) $(P(x), Q(x))$ is either of the form $(\pm(x^2 - 4x) + a, \pm(x^2 - 5x) + b)$, where the $\pm$'s are independent (so there are four "classes" of solutions).

Next, note that Q is symmetric around $x = \frac{5}{2}$, so $P(Q(0)) = P(Q(5))$. Equating the right-hand sides gives $P(0) \cdot 7 = P(5) \cdot 2$, or $7a = 2(\pm 5 + a)$, so $a = \pm 2$.

Similarly, P is symmetric around $x = 2$, so $Q(P(0)) = Q(P(4))$. Equating the right-hand sides gives $Q(0) \cdot (-2) = Q(4) \cdot 2$, or $-b = \mp 4 + b$, so $b = \pm 2$.

This narrows us down to $(P(x), Q(x))$ is $(\pm(x^2 - 4x + 2), \pm(x^2 - 5x + 2))$. To figure out which combination of signs works, we plug in 0 and see if the equations work. We first compute $P(0) = \pm 2$ and $Q(0) = \pm 2$, then $P(\pm 2) = 6 \mp 8$ and $Q(\pm 2) = 6 \mp 10$.

Note that these are all integers, and the RHS of the first equation is divisible by 7. This means $7 \mid P(Q(0))$, so $P(Q(0)) = 14$ and thus $Q(0) = -2$. This means P has positive leading coefficient while Q has negative leading coefficient. This narrows down to only one possibility: $P(x) = x^2 - 4x + 2$ and $Q(x) = -x^2 + 5x - 2$, which one can see does in fact work for all x. This gives $P(10) + Q(-10) = \boxed{-90}$.

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11. For $0 \leq x \leq 2\pi$, $f(x) = \frac{1}{2}\left|\sin(20x) + \cos(24x)\right|$ is always strictly between 0 and 1. Estimate the average value of f over this interval. Express your answer in the form $0.abcdef$.

Proposed by Connor Gordon

Answer. 0.407759

Solution.

1

$$f(x) = \frac{1}{2}\left|\sin(20x) + \cos(24x)\right|$$

2

$$\frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$$

$= 0.407759276879$