Algebra and Number Theory Round

Instructions

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1. Do not look at the test before the proctor starts the round.
2. This test consists of 10 short-answer problems to be solved in 50 minutes. The final estimation question will be used to break ties.
3. No computational aids other than pencil/pen are permitted.
4. Write your name and team name on your answer sheet.
5. Write your answers in the corresponding lines on your answer sheet.
6. Answers must be reasonably simplified.
7. If you believe that the test contains an error, submit your protest to the 2024 CMIMC discord.

Algebra and Number Theory

1. Connor is thinking of a two-digit number n, which satisfies the following properties:

   - If $n > 70$, then n is a perfect square.
   - If $n > 40$, then n is prime.
   - If $n < 80$, then the sum of the digits of n is 14.

   What is Connor's number?

2. Suppose $P(x) = x^2 + Ax + B$ for real A and B. If the sum of the roots of $P(2x)$ is $\frac{1}{2}$ and the product of the roots of $P(3x)$ is $\frac{1}{3}$, find $A + B$.

3. The positive integer 8833 has the property that $8833 = 88^2 + 33^2$. Find the (unique) other four-digit positive integer $\overline{abcd}$ where $\overline{abcd} = (\overline{ab})^2 + (\overline{cd})^2$.

4. For positive integer n, let $f(n)$ be the largest integer k such that $k! \leq n$, let $g(n) = n - (f(n))!$, and for $j \geq 1$ let
$$g^j(n) = \underbrace{g(\dots(g(n))\dots)}_{j \text{ times}}.$$
   Find the smallest positive integer n such that $g^j(n) > 0$ for all $j < 30$ and $g^{30}(n) = 0$.

5. Let
$$f(x) = (x+1)^6 + (x-1)^5 + (x+1)^4 + (x-1)^3 + (x+1)^2 + (x-1)^1 + 1.$$
   Find the remainder when $\sum_{j=-126}^{126} jf(j)$ is divided by 1000.

6. Integers a, b satisfy the following property: the line $y = 2x + ab$ passes through all intersection points of the two parabolas given by
$$y = x^2 + 2x + a, \quad y = 2x^2 + bx,$$
   which intersect at least once. How many such (a, b) satisfy $|ab| \leq 100$?

7. Let x_0, x_1, x_2, and x_3 be complex numbers forming a square centered at 0 in the complex plane with side length 2. For each $0 \leq k \leq 3$, there are four more complex numbers $z_{4k}, z_{4k+1}, z_{4k+2}$, and z_{4k+3} forming a square centered at x_k with side length $\sqrt{2}$. Given that $\prod_{i=0}^{15} z_i$ is a positive integer, how many possible values could it take?

8. Compute the number of non-negative integers $k < 2^{20}$ such that $\binom{5k}{k}$ is odd.

9. Let $\mathbb{Q}_{\geq 0}$ be the non-negative rational numbers, $f : \mathbb{Q}_{\geq 0} \to \mathbb{Q}_{\geq 0}$ such that $f(z+1) = f(z) + 1$, $f(1/z) = f(z)$ for $z \neq 0$, and $f(0) = 0$. Define a sequence P_n of non-negative integers recursively via
$$P_0 = 0, \quad P_1 = 1, \quad P_n = 2P_{n-1} + P_{n-2}$$
   for every $n \geq 2$. Find $f\left(\frac{P_{20}}{P_{24}}\right)$.

10. There exists a unique pair of polynomials $(P(x), Q(x))$ such that
$$P(Q(x)) = P(x)(x^2 - 6x + 7)$$
$$Q(P(x)) = Q(x)(x^2 - 3x - 2)$$
    Compute $P(10) + Q(-10)$.

11. **(Tiebreaker)** For $0 \leq x \leq 2\pi$, $f(x) = \frac{1}{2}|\sin(20x) + \cos(24x)|$ is always strictly between 0 and 1. Estimate the average value of f over this interval. Express your answer in the form $0.abcdef$.