CMIMC 2023

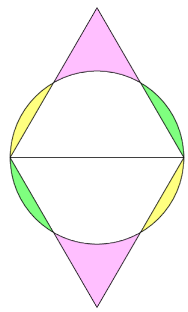
Team Round Solutions

1. On a plane, two equilateral triangles (of side length 1) share a side, and a circle is drawn with the common side as a diameter. Find the area of the set of all points that lie inside exactly one of these shapes.

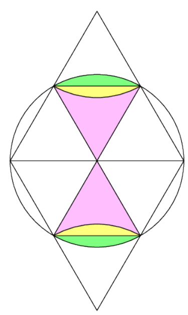
   Proposed by Howard Halim

   Answer. $\frac{\pi}{12}$

   Solution. We need to calculate the total colored area:



By moving the regions to these positions, we see that they cover a third of the circle:



The circle has radius $\frac{AB}{2} = \frac{1}{2}$, so its area is $\frac{\pi}{4}$, and the answer is $\boxed{\frac{\pi}{12}}$

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2. Real numbers x and y satisfy

$$x^2 + y^2 = 2023$$
$$(x-2)(y-2) = 3.$$

   Find the largest possible value of $|x - y|$.

   Proposed by Howard Halim

   Answer. $\sqrt{2197}$

   Solution. Note it suffices to minimize $(x-y)^2 = (x+y)^2 - 4xy$. Letting $s = x+y$ and $p = xy$, the given equations rearrange to $s^2 - 2p = 2023$ and $p - 2s = -1$. Substituting $p = 2s - 1$ into the first equation gives $s^2 - 4s = 2021 \to (s-2)^2 = 2025 \to s = -43, 47$. These correspond to $p = -87$ and $p = 93$ respectively.

We wish to minimize $s^2 - 4p$. We compute $(-43)^2 - 4(-87) = 2197$ and $47^2 - 4(93) = 1837$. The larger of these is 2197, yielding an answer of $\boxed{\sqrt{2197}}$.

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3. Find the number of ordered triples of positive integers (a, b, c), where $1 \le a, b, c \le 10$, with the property that $\gcd(a, b), \gcd(a, c)$, and $\gcd(b, c)$ are all pairwise relatively prime.

   Proposed by Kyle Lee

   Answer. 841

   Solution. Let $a = 2^{a_1} \cdot 3^{a_2} \cdots, b = 2^{b_1} \cdot 3^{b_2} \cdots$, and $c = 2^{c_1} \cdot 3^{c_2} \cdots$. By the definition fo the gcd, the condition on the exponents of the factors of 2 is equivalent to

$$\min(\min(a_1, b_1), \min(b_1, c_1)) = 0$$
$$\min(\min(a_1, b_1), \min(a_1, c_1)) = 0$$
$$\min(\min(a_1, c_1), \min(b_1, c_1)) = 0,$$

   which implies at least one of a_1, b_1, c_1 must be 0. Consider the complement; that is, when none of the exponents on the factors of 2 are 0. Since there are 5 multiples of 2, 3 multiples of 3, 2 multiples of 5, and 1 multiple of 7 in the given range, we have an initial count of $5^3 + 3^3 + 2^3 + 1^3$ such triples. However, we overcount the two triples $(2\cdot3, 2\cdot3, 2\cdot3)$ and $(2\cdot5, 2\cdot5, 2\cdot5)$, so there are actually $5^3 + 3^3 + 2^3 + 1^3 - 2 = 159$ ordered triples. Since this is the complement, it follows that the desired answer is $10^3 - 159 = \boxed{841}$.

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4. Suppose $a_1, a_2, a_3, \ldots,$ is a sequence of real numbers such that

$$a_n = \frac{a_{n-1}a_{n-2}}{3a_{n-2} - 2a_{n-1}}$$

   for all $n \ge 3$. If $a_1 = 1$ and $a_{10} = 10$, what is a_{19}?

   Proposed by Howard Halim

   Answer. $-\frac{10}{4607}$

   Solution. We can rewrite the equation as

$$a_n = \frac{1}{\frac{3}{a_{n-1}} - \frac{2}{a_{n-2}}} \iff \frac{1}{a_n} = \frac{3}{a_{n-1}} - \frac{2}{a_{n-2}} \iff \frac{1}{a_n} - \frac{1}{a_{n-1}} = 2\left(\frac{1}{a_{n-1}} - \frac{1}{a_{n-2}}\right)$$

   Let $d = \frac{1}{a_2} - \frac{1}{a_1}$, then we have

$$\frac{1}{a_{10}} - \frac{1}{a_1} = (2^9 - 1)d \implies d = -\frac{9}{10(2^9 - 1)}$$

and

$$\frac{1}{a_{19}} - \frac{1}{a_1} = (2^{18}-1)d \implies a_{19} = \boxed{-\frac{10}{4607}}$$

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5. 1296 CMU Students sit in a circle. Every pair of adjacent students rolls a standard six-sided die, and the 'score' of any individual student is the sum of their two dice rolls. A 'matched pair' of students is an (unordered) pair of distinct students with the same score. What is the expected value of the number of matched pairs of students?

   Proposed by Dilhan Salgado

   Answer. 94605

   Solution. Let $n = 1296$.

   There are n adjacent pairs of students. Each adjacent pair will be matched iff their other dice roll is the same, which happens with probability $\frac{1}{6}$. Thus the expected number of matched adjacent pairs is $\frac{n}{6}$.

   There are $\frac{n(n-3)}{2}$ non-adjacent pairs of students. Each non-adjacent pair's scores are completely independent. Thus, the probability of them having the same score is:

$$\frac{1\cdot 1 + 2\cdot 2 + ... + 5\cdot 5 + 6\cdot 6 + 5\cdot 5 + ... + 1\cdot 1}{6^4}$$

   Computing this out we get $\frac{1+4+9+16+25+36+25+16+9+4+1}{1296} = \frac{55+91}{1296} = \frac{146}{1296}$. This means the expected number of non-adjacent matched pairs is $\frac{73n(n-3)}{1296}$.

   Adding these up, we get that the expected value of total matched pairs is:

$$\frac{1296}{6} + \frac{73\cdot 1293\cdot 1296}{1296} = 216 + 73\cdot 1293 = \boxed{94605}$$

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6. A positive integer n is said to be base-able if there exists positive integers a and b, with $b > 1$, such that $n = a^b$. How many positive integer divisors of 729000000 are base-able?

   Proposed by Kyle Lee

   Answer. 90

   Solution. Note that $729000000 = 30^6 = 2^6 \cdot 3^6 \cdot 5^6$. First, observe that 1 is trivially base-able as 1^b works for any $b > 1$. Now, we case on the value of b. Clearly, $2 \le b \le 6$. If $b = 2$, then just focusing on the factors of 2 in a, we can have either $2^0, 2^1, 2^2$, or 2^3. Hence, across all prime factors, we have $4^3 - 1$ possibilities, where we subtract 1 for the case when $n = 1$. If $b = 3$, then again focusing on the factors of 2 in a, we can have either $2^0, 2^1$, or 2^2, for $3^3 - 1$ possibilities. For $b = 4, 5$, and 6, we can only have 2^0 or 2^1, for a total of $2^3 - 1$ possibilities each. Hence, we have an initial count of

$$1 + (4^3 - 1) + (3^3 - 1) + 3(2^3 - 1) = 111.$$

However, some of these values are overcounted. Focusing on just the factors of 2, we can easily see that $2^4 = (2^2)^2$ and $2^6 = (2^2)^3 = (2^3)^2$ are the only overcounted possibilities. For the first case, we can simply subtract off $2^3 - 1$ by ignoring when $b = 4$. However, for the second case, if we subtract off $2^3 - 1$ by ignoring when $b = 6$, we still overcount the cases when $b = 2$ and $b = 3$. Hence, we have to subtract $2^3 - 1$ again to account for cases such as $(2^2)^3 = (2^3)^2$. Hence, the answer is $111 - 3(2^3 - 1) = \boxed{90}$.

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7. Compute the value of
$$\sin^2\left(\frac{\pi}{7}\right) + \sin^2\left(\frac{3\pi}{7}\right) + \sin^2\left(\frac{5\pi}{7}\right).$$
Your answer should not involve any trigonometric functions.

Proposed by Howard Halim

Answer. $\frac{7}{4}$

Solution. First note $\sin^2(\frac{5\pi}{7}) = \sin^2(\frac{2\pi}{7})$. Now define
$$S = \sin^2\left(\frac{\pi}{7}\right) + \sin^2\left(\frac{3\pi}{7}\right) + \sin^2\left(\frac{5\pi}{7}\right)$$
$$C = \cos^2\left(\frac{\pi}{7}\right) + \cos^2\left(\frac{3\pi}{7}\right) + \cos^2\left(\frac{5\pi}{7}\right)$$
We can easily compute $C + S = 3$ by the identity $\sin^2(\theta) + \cos^2(\theta) = 1$, and $C - S = \cos(\frac{2\pi}{7}) + \cos(\frac{4\pi}{7}) + \cos\left(\frac{6\pi}{7}\right)$ by the identity $\cos^2(\theta) - \sin^2(\theta) = \cos(2\theta)$. It remains to compute this latter value. Letting $\omega = e^{2\pi/7}$ (a seventh root of unity), we have
$$\begin{aligned}\cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{6\pi}{7}\right) &= \frac{1}{2}(\omega + \omega^{-1}) + \frac{1}{2}(\omega^2 + \omega^{-2}) + \frac{1}{2}(\omega^3 + \omega^{-3}) \\ &= \frac{1}{2}(\omega + \omega^6 + \omega^2 + \omega^5 + \omega^3 + \omega^4).\end{aligned}$$
Recalling that ω is a root of $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$, we see that this is simply equal to $-\frac{1}{2}$. Solving the system $C + S = 3$, $C - S = -\frac{1}{2}$ yields $S = \boxed{\frac{7}{4}}$.

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8. NASA is launching a spaceship at the south pole, but a sudden earthquake shock caused the spaceship to be launched at an angle of θ from vertical ($0 < \theta < 90°$). The spaceship crashed back to Earth, and NASA found the debris floating in the ocean in the northern hemisphere. NASA engineers concluded that $\tan\theta > M$, where M is maximal. Find M.

Assume that the Earth is a sphere, and the trajectory of the spaceship (in the reference frame of Earth) is an ellipse with the center of the Earth one of the foci.

Proposed by Kevin You

Answer. $\sqrt{2} - 1$

Solution. Let O be the center of Earth, F be the second focus, A be the south pole, and B the location of the debris. The ellipse possesses the property that the normal to the ellipse at A (A is any point on the ellipse) bisects $\angle OAF$. Since the normal is perpendicular to the initial velocity $\vec{v}$, and the angle between $\vec{v}$ and OA is θ, we can conclude that $\angle OAF = 180^\circ - 2\theta$.

Since $OA = OB$, the situation is symmetric, and we also have $\angle OBF = 180^\circ$. Finally, we know that $\angle AOB > 90^\circ$. The three relations combined gives $4\theta > 90^\circ + \angle AFB$. By taking F to be very far away, we have that $\angle AFB \to 0^\circ$, and so $\theta > 22.5^\circ$ is tight. This gives $M = \boxed{\sqrt{2}-1}$.

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9. A positive integer N is a *triple-double* if there exists non-negative integers a, b, c such that $2^a + 2^b + 2^c = N$. How many three-digit numbers are triple-doubles?

   Proposed by Giacomo Rizzo

   Answer. 115

   Solution. The set of triple-doubles is the set of positive integers greater than 2 whose binary representation contains at most three ones. Since the binary representation of a three-digit triple-double is between $100_{10} = 1100100_2$ and $999_{10} = 1111100111_2$ inclusive, it must have between 7 and 10 binary digits. There are $\binom{10}{3}+\binom{10}{2}+\binom{10}{1}+\binom{10}{0} = 176$ ways to select at most three binary digits to be ones, and $\binom{7}{3}+\binom{7}{2}+\binom{7}{1}+\binom{7}{0} = 64$ such combinations produce a number with seven or fewer binary digits. Since 1100100_2, 1101000_2, and 1110000_2 are the only valid seven-digit binary representations, our answer is $176 - 64 + 3 = \boxed{115}$.

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10. Consider the set of all permutations, $\mathcal{P}$, of $\{1, 2, \ldots, 2022\}$. For permutation $P \in \mathcal{P}$, let P_1 denote the first element in P. Let $\operatorname{sgn}(P)$ denote the sign of the permutation. Compute the following number modulo 1000:

$$\sum_{P\in\mathcal{P}} \frac{P_1 \cdot \operatorname{sgn}(P)^{P_1}}{2020!}.$$

   (The *sign* of a permutation P is $(-1)^k$, where k is the minimum number of two-element swaps needed to reach that permutation).

   Proposed by Nairit Sarkar

   Answer. 772

   Solution. Take 2020! out of sum and divide at end. Casework on P_1 odd or even.

   Assume P_1 odd for the first case. We wish to compute

$$\sum_{P\in\mathcal{P}} P_1 \cdot \operatorname{sgn}(P).$$

   Let S be the set of permutations with $P_1 = i$ for odd i. Swapping the second and third elements yields a bijection between the odd and even elements in S. Therefore, this sum is 0.

Assume P_1 even. Then, we wish to compute

$$\sum_{P \in \mathcal{P}} P_1.$$

There are $(n-1)!$ permutations starting with each even integer from 2 to 2022 inclusive. Therefore, our answer is $1011 \cdot 1012 \cdot 2021!$. Dividing by $2020!$ gives $1011 \cdot 1012 \cdot 2021$. We see that is $\boxed{772}$ modulo 1000.

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11. A positive integer is *detestable* if the sum of its digits is a multiple of 11. How many positive integers below 10000 are detestable?

Proposed by Giacomo Rizzo

Answer. $\boxed{908}$

Solution. All detestable positive integers below 10000 can be written in the form $x = \overline{abcd}$ (where a, b, c, d are nonnegative digits), so we wish to find all such (a, b, c, d) satisfying $a+b+c+d \neq 0$ and $a+b+c+d \equiv 0 \pmod{11}$. Since $0 < a+b+c+d \leq 36$, there are three cases for $a+b+c+d$.

Case 1: $a + b + c + d = 11$. By balls and boxes counting, there are $\binom{11+4-1}{4-1} = 364$ nonnegative solutions. However, since a, b, c, d are digits, they cannot be greater than 9. If one box has 10 or more balls, there are 4 ways to select the box and $\binom{1+4-1}{4-1}$ ways to distribute the remaining ball across 4 boxes, for a total of 16 ways. So, there are $364 - 16 = 348$ solutions.

Case 2: $a + b + c + d = 22$. By balls and boxes counting, there are $\binom{22+4-1}{4-1} = 2300$ nonnegative solutions. Once again, we need to subtract invalid solutions. If one box has 10 or more balls, there are 4 ways to select the box and $\binom{12+4-1}{4-1} = 455$ ways to distribute the remaining 12 balls across 4 boxes, for a total of 1820 ways. But per PIE, we need to add back the solutions where 2 boxes have 10 or more balls. There are $\binom{4}{2} = 6$ ways to select the boxes and $\binom{2+4-1}{4-1} = 10$ ways to distribute the remaining 2 balls across 4 boxes, for a total of 60 ways. So, there are $2300 - 1820 + 60 = 540$ solutions.

Case 3: $a + b + c + d = 33$. There are not many possibilities that a, b, c, d can be, as they cannot be greater than 9. Listing them out, we get $(a, b, c, d) = (9, 9, 9, 6)$, $(9, 9, 8, 7)$, $(9, 8, 8, 8)$ and their reorderings, for a total of $4 + 12 + 4 = 20$ solutions.

It follows from our casework that there are $348 + 540 + 20 = \boxed{908}$ detestable positive integers below 10000.

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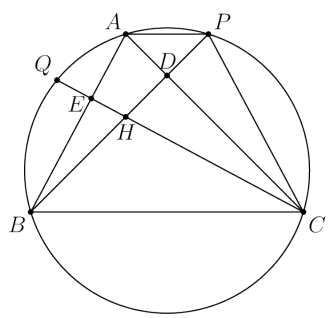
12. Let ABC be an acute triangle with circumcircle ω. Let D and E be the feet of the altitudes from B and C onto sides AC and AB, respectively. Lines BD and CE intersect ω again at points $P \neq B$ and $Q \neq C$. Suppose that $PD = 3$, $QE = 2$, and $AP \parallel BC$. Compute DE.

Proposed by Kyle Lee

Answer. $\sqrt{2} + \sqrt{7}$

Solution. First, note that the reflection of P and Q across AC and AB coincide at $H = BD \cap CE$, the orthocenter of ABC. Hence, $HD = 3$ and $HE = 2$. Now, since $AP \parallel BC$, it follows that $APCB$ is an isosceles trapezoid, so $BD = CD$. Hence, $\angle ACB = 45°$, which implies $\angle HAD = 45°$. Therefore, $AD = 3$ and $AH = 3\sqrt{2}$. By the Pythagorean Theorem, it follows that $AE^2 = (3\sqrt{2})^2 - 2^2$, so $AE = \sqrt{14}$. Lastly, since $\angle AEH + \angle ADH = 180°$, it follows that $ADHE$ is cyclic, so by Ptolemy's theorem, we have $2(3) + 3(\sqrt{14}) = (DE)(3\sqrt{2})$, so $DE = \boxed{\sqrt{2} + \sqrt{7}}$.



13. Suppose that the sequence of real numbers $a_1, a_2, \ldots$ satisfies $a_1 = -\sqrt{1}, a_2 = \sqrt{2}$, and for all $k > 1$,

$$\frac{a_{k+1} + a_{k-1}}{a_k} = \frac{\sqrt{3} + \sqrt{1}}{\sqrt{2}}.$$

Find a_{2023}.

Proposed by Kevin You

Answer. $4 + 3\sqrt{3}$

Solution. With wishful thinking, we may hope that the sequence resembles a geometric sequence, since only ratios are involved. Take $a_k = e^{\alpha k}$. We get

$$\frac{e^{k\alpha + \alpha} + e^{k\alpha - \alpha}}{e^{k\alpha}} = e^{\alpha} + e^{-\alpha} = \frac{\sqrt{3} + \sqrt{1}}{\sqrt{2}}.$$

Unfortunately, this is not solvable in real numbers for α. However, if we allow complex numbers, then take $\alpha = i\theta$. We have that

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta = \frac{\sqrt{3} + \sqrt{1}}{\sqrt{2}}$$

We are in luck, as $\theta = 15°$ is a solution. Let us write the sequence as a trigonometric function. Both $\cos(k\theta)$ and $\sin(k\theta)$ satisfies the given recurrence. Indeed, by sum to product we have

$$\frac{\sin((k+1)\theta) + \sin((k-1)\theta)}{\sin(k\theta)} = \frac{2\sin(k\theta)\sin(\theta)}{\sin(k\theta)} = 2\sin(\theta).$$

The case for cos is similar. So, we write

$$a_k = A_1\cos(k\theta) + A_2\sin(k\theta)$$

By shifting the sequence by 1 and fitting the initial conditions, we find that

$$a_k = -\cos((k-1)\theta) + \left(4 + 3\sqrt{3}\right)\sin((k-1)\theta)$$

So, we get

$$a_{2023} = \boxed{4 + 3\sqrt{3}}$$

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14. Let ABC be points such that $AB = 7, BC = 5, AC = 10$, and M be the midpoint of AC. Let ω, ω_1 be the circumcircles of ABC and BMC. Ω, Ω_1 are circles through A and M such that Ω is tangent to ω_1 and Ω_1 is tangent to the line through the centers of ω_1 and Ω. D, E be the intersection of Ω with ω and Ω_1 with ω_1. If F is the intersection of the circumcircle of DME with BM, find FB.

Proposed by David Tang

Answer. $4\sqrt{3}$

Solution. Invert about M, denote inverted points X by X'. Then, we find that D' is the intersection of the line parallel to $B'C'$ through A' and the circumcircle of $A'B'C'$. E' is the foot of the perpendicular dropped from A' to $B'C'$. F' is the intersection of $D'E'$ with $B'M$. If N is the midpoint of $B'C'$, then we notice that $D'E'$ cuts AN into a ratio of $2 : 1$. Thus, it passes through the centroid of $A'B'C'$, so F' is the centroid of $A'B'C'$ since it is on $D'E'$ and the B'-median. Thus, $FB/BM = F'B'/F'M = 2$. Since $BM = 2\sqrt{3}$ from the median length formula, we get that $FB = 4\sqrt{3}$.

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15. Equilateral triangle T_0 with side length 3 is on a plane. Given triangle T_n on the plane, triangle T_{n+1} is constructed on the plane by translating T_n by 1 unit, in one of six directions parallel to one of the sides of T_n. The direction is chosen uniformly at random.

Let a be the least integer such that at most one point on the plane is in or on all of $T_0, T_1, T_2, \ldots, T_a$. It can be shown that a exists with probability 1. Find the probability that a is even.

Proposed by Justin Hsieh

Answer. $\frac{9}{22}$

Solution. This problem uses states and lots of infinite series using the Fibonacci sequence.

After placing T_n, we can be in one of five states. We will name the states $\mathcal{S}, \mathcal{N}, \mathcal{V}, \mathcal{E}, \mathcal{X}$, based on the intersection of $T_0, T_1, T_2, \ldots, T_n$, and the placement of that set in relation to T_n. $\mathcal{S}$ ("Start"): the intersection is an equilateral triangle of side length 3, $\mathcal{N}$ ("Next"): the intersection is an equilateral triangle of side length 2, $\mathcal{V}$ ("Vertex"): the intersection is an equilateral triangle of side length 1, adjacent to two sides of T_n, $\mathcal{E}$ ("Edge"): the intersection is an equilateral triangle of side length 1, adjacent to one side of T_n, $\mathcal{X}$: the intersection is a single point.

We may move to a different state after placing T_{n+1}, given T_n. We start at $\mathcal{S}$, and move to $\mathcal{N}$ with probability 1. At $\mathcal{N}$, there is a $\frac{1}{3}$ chance of staying at $\mathcal{N}$, a $\frac{1}{3}$ chance of moving to $\mathcal{V}$, and a $\frac{1}{3}$ chance of moving to $\mathcal{E}$. At $\mathcal{V}$, there is a $\frac{1}{3}$ chance of moving to $\mathcal{E}$, and a $\frac{2}{3}$ chance of moving to $\mathcal{X}$. At $\mathcal{E}$, there is a $\frac{1}{3}$ chance of staying at $\mathcal{E}$, a $\frac{1}{3}$ chance of moving to $\mathcal{V}$, and a $\frac{1}{3}$ chance of moving to $\mathcal{X}$.

If we define O_s to be the probability of taking an odd number of steps to reach the end from state s, we have $O_X = 0$, $O_V = 9/11$, $O_E = 6/11$, $O_N = 9/22$. Since we get to state N in 1 step, the answer is the odd probability of state N which is $\boxed{9/22}$.