CMIMC 2023

Team Round

Instructions

1. Do not look at the test before the proctor starts the round.
2. This test consists of 15 short-answer problems to be solved in 60 minutes.
3. No computational aids other than pencil/pen are permitted.
4. Write your team name on your answer sheet.
5. Write your answers in the corresponding lines on your answer sheet.
6. Answers must be reasonably simplified.
7. If you believe that the test contains an error, submit your protest to the 2023 CMIMC discord.

Team

1. On a plane, two equilateral triangles (of side length 1) share a side, and a circle is drawn with the common side as a diameter. Find the area of the set of all points that lie inside exactly one of these shapes.

2. Real numbers x and y satisfy

$$x^2 + y^2 = 2023$$
$$(x-2)(y-2) = 3.$$

Find the largest possible value of $|x - y|$.

3. Find the number of ordered triples of positive integers (a, b, c), where $1 \leq a, b, c \leq 10$, with the property that $\gcd(a, b), \gcd(a, c)$, and $\gcd(b, c)$ are all pairwise relatively prime.

4. Suppose $a_1, a_2, a_3, \ldots,$ is a sequence of real numbers such that

$$a_n = \frac{a_{n-1}a_{n-2}}{3a_{n-2} - 2a_{n-1}}$$

for all $n \geq 3$. If $a_1 = 1$ and $a_{10} = 10$, what is a_{19}?

5. 1296 CMU Students sit in a circle. Every pair of adjacent students rolls a standard six-sided die, and the 'score' of any individual student is the sum of their two dice rolls. A 'matched pair' of students is an (unordered) pair of distinct students with the same score. What is the expected value of the number of matched pairs of students?

6. A positive integer n is said to be base-able if there exists positive integers a and b, with $b > 1$, such that $n = a^b$. How many positive integer divisors of 729000000 are base-able?

7. Compute the value of

$$\sin^2\left(\frac{\pi}{7}\right) + \sin^2\left(\frac{3\pi}{7}\right) + \sin^2\left(\frac{5\pi}{7}\right).$$

Your answer should not involve any trigonometric functions.

8. NASA is launching a spaceship at the south pole, but a sudden earthquake shock caused the spaceship to be launched at an angle of θ from vertical ($0 < \theta < 90°$). The spaceship crashed back to Earth, and NASA found the debris floating in the ocean in the northern hemisphere. NASA engineers concluded that $\tan\theta > M$, where M is maximal. Find M.

   Assume that the Earth is a sphere, and the trajectory of the spaceship (in the reference frame of Earth) is an ellipse with the center of the Earth one of the foci.

9. A positive integer N is a *triple-double* if there exists non-negative integers a, b, c such that $2^a + 2^b + 2^c = N$. How many three-digit numbers are triple-doubles?

10. Consider the set of all permutations, $\mathcal{P}$, of $\{1, 2, \ldots, 2022\}$. For permutation $P \in \mathcal{P}$, let P_1 denote the first element in P. Let $\operatorname{sgn}(P)$ denote the sign of the permutation. Compute the following number modulo 1000:

$$\sum_{P \in \mathcal{P}} \frac{P_1 \cdot \operatorname{sgn}(P)^{P_1}}{2020!}.$$

(The *sign* of a permutation P is $(-1)^k$, where k is the minimum number of two-element swaps needed to reach that permutation).

11. A positive integer is *detestable* if the sum of its digits is a multiple of 11. How many positive integers below 10000 are detestable?

12. Let ABC be an acute triangle with circumcircle ω. Let D and E be the feet of the altitudes from B and C onto sides AC and AB, respectively. Lines BD and CE intersect ω again at points $P \neq B$ and $Q \neq C$. Suppose that $PD = 3$, $QE = 2$, and $AP \parallel BC$. Compute DE.

13. Suppose that the sequence of real numbers $a_1, a_2, \ldots$ satisfies $a_1 = -\sqrt{1}, a_2 = \sqrt{2}$, and for all $k > 1$,

$$\frac{a_{k+1} + a_{k-1}}{a_k} = \frac{\sqrt{3} + \sqrt{1}}{\sqrt{2}}.$$

Find a_{2023}.

14. Let ABC be points such that $AB = 7, BC = 5, AC = 10$, and M be the midpoint of AC. Let ω, ω_1 be the circumcircles of ABC and BMC. Ω, Ω_1 are circles through A and M such that Ω is tangent to ω_1 and Ω_1 is tangent to the line through the centers of ω_1 and Ω. D, E be the intersection of Ω with ω and Ω_1 with ω_1. If F is the intersection of the circumcircle of DME with BM, find FB.

15. Equilateral triangle T_0 with side length 3 is on a plane. Given triangle T_n on the plane, triangle T_{n+1} is constructed on the plane by translating T_n by 1 unit, in one of six directions parallel to one of the sides of T_n. The direction is chosen uniformly at random.

    Let a be the least integer such that at most one point on the plane is in or on all of $T_0, T_1, T_2, \ldots, T_a$. It can be shown that a exists with probability 1. Find the probability that a is even.