Geometry Round Solutions

1. Triangle ABC is isosceles with $AB = AC$. The bisectors of angles ABC and ACB meet at I. If the measure of angle CIA is $130°$, compute the measure of angle CAB.

   Proposed by Connor Gordon

   Answer. $20°$

   Solution. By symmetry, $m\angle BIA = 130°$ as well, so $m\angle BIC = (360 - 130 - 130)° = 100°$. Then $m\angle IBC = m\angle ICB = \frac{1}{2}(180 - 100)° = 40°$, so $m\angle ABC = m\angle ACB = (2 \cdot 40)° = 80°$, and finally $m\angle CAB = (180 - 80 - 80)° = \boxed{20°}$.

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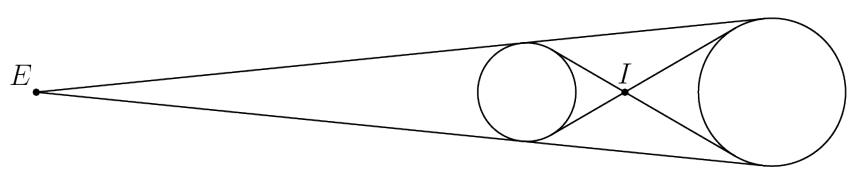
2. Two circles have radius 2 and 3, and the distance between their centers is 10. Let E be the intersection of their two common external tangents, and I be the intersection of their two common internal tangents. Compute EI.

   Proposed by Connor Gordon

   Answer. 24

   Solution. For convenience, place everything on a number line with the centers of the circles at 0 (for the circle of radius 2) and 10 (for the circle of radius 3. Notice that if we look at the points of tangency from E to each circle, we get similar triangles with a similarity ratio of $2 : 3$. Moreover, E is on the "same side" of both circles, so in particular $\frac{0-E}{10-E} = \frac{2}{3}$, which yields $E = -20$.

   The same reasoning applies to I, except it's on "different sides" of the circle, so we have $\frac{10-I}{I-0} = \frac{2}{3}$, or $I = 4$. Then $EI = |-20 - 4| = \boxed{24}$.



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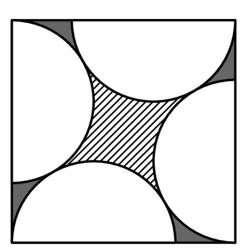
3. Four semicircles of radius 1 are placed in a square, as shown below. The diameters of these semicircles lie on the sides of the square and each semicircle touches a vertex of the square. Find the absolute difference between the shaded area and the "hatched" area.

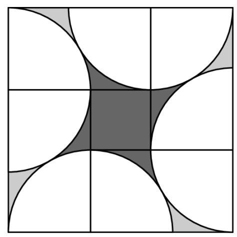


Proposed by Connor Gordon

Answer. $4 - 2\sqrt{3}$

Solution. Divide the square into four rectangles and a smaller square as shown. By symmetry, the light and dark regions in each rectangle have the same area, so the only remaining difference is the square in the middle.

We can see the rectangles have short side 1 and diagonal 2, so the long side must be $\sqrt{3}$. Thus the side length of the small square is $\sqrt{3} - 1$, so its area is $\boxed{4 - 2\sqrt{3}}$.



4. A rhombus $\mathcal{R}$ has short diagonal of length 1 and long diagonal of length 2023. Let $\mathcal{R}'$ be the rotation of $\mathcal{R}$ by 90° about its center. If $\mathcal{U}$ is the set of all points contained in either $\mathcal{R}$ or $\mathcal{R}'$ (or both; this is known as the *union* of $\mathcal{R}$ and $\mathcal{R}'$) and $\mathcal{I}$ is the set of all points contained in both $\mathcal{R}$ and $\mathcal{R}'$ (this is known as the *intersection* of $\mathcal{R}$ and $\mathcal{R}'$, compute the ratio of the area of $\mathcal{I}$ to the area of $\mathcal{U}$.

   Proposed by Connor Gordon

   Answer. $\frac{1}{2023}$

   Solution. Draw lines from the center to each of the intersection points to split the union into four congruent kites. The area of the portion of the kite inside the intersection is $\frac{1}{2} \cdot \frac{1}{2} \cdot h$ for some h that I don't really care to calculate. The area of the entire kite is $\frac{1}{2} \cdot \frac{2023}{2} \cdot h$ for the same h, so the ratio is simply $\boxed{\frac{1}{2023}}$.

5. In trapezoid $ABCD, AB = 3, BC = 2, CD = 5$, and $\angle B = \angle C = 90^\circ$. The angle bisectors of $\angle A$ and $\angle D$ intersect at a point P in the interior of $ABCD$. Compute $BP^2 + CP^2$.

   Proposed by Kyle Lee

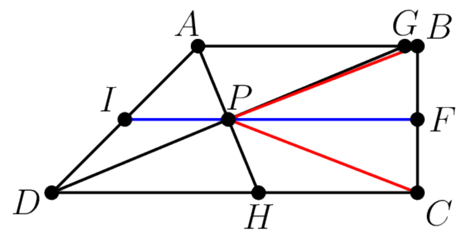
   Answer. $38 - 16\sqrt{2}$

   Solution. Let $G = AB \cap DP$ and $H = CD \cap AP$. First, observe that $\angle APD = 90^\circ$ since $\angle BAD + \angle ADC = 180^\circ$. Now, since $\angle GAP = \angle DAP$, we have $AD = AG$. A similar argument reveals $AD = HD$, so $ADHG$ is a rhombus. Then, by symmetry, we have $BP = CP$. Let the line passing through P, perpendicular to BC, intersect AD and BC at I and F, respectively. Then,

$BP^2 + CP^2 = 2BP^2 = 2(1 + FP^2)$. But,

$$FP = FI - PI = \frac{3+5}{2} - \frac{\sqrt{2^2 + (5-3)^2}}{2} = 4 - \sqrt{2},$$

so $BP^2 + CP^2 = 2(1 + (4 - \sqrt{2})^2) = \boxed{38 - 16\sqrt{2}}$.



6. Let $ABCD$ be a regular tetrahedron. Suppose points X, Y, and Z lie on rays AB, AC, and AD respectively such that $XY = YZ = 7$ and $XZ = 5$. Moreover, the lengths AX, AY, and AZ are all distinct. Find the volume of tetrahedron $AXYZ$.

   Proposed by Connor Gordon

   Answer. $\sqrt{122}$

   Solution. Let $x = AX$, $y = AY$, and $z = AZ$ for brevity. The desired volume is $\frac{\sqrt{2}}{12}xyz$. The Law of Cosines on each of $\triangle AXY$, $\triangle AYZ$, and $\triangle AZX$ yields

$$x^2 + y^2 - xy = 49$$
$$y^2 + z^2 - yz = 49$$
$$x^2 + z^2 - xz = 25$$

   Subtracting the second equation from the first gives

$$x^2 - z^2 - xy + yz = 0 \implies (x + z - y)(x - z) = 0.$$

   By hypothesis $x \neq z$, so $x + z = y$. Substituting this into the first equation gives

$$x^2 + (x+z)^2 - x(x+z) = 49 \to x^2 + z^2 + xz = 49$$

   Subtracting the third equation from this gives $2xz = 24 \implies xz = 12$. We can then rewrite the third equation as

$$(x+z)^2 - 3xz = 25 \to x + z = \sqrt{61}.$$

   The desired answer is then

$$\frac{\sqrt{2}}{12}xyz = \frac{\sqrt{2}}{12}xz(x+z) = \frac{\sqrt{2}}{12}(12)(\sqrt{61}) = \boxed{\sqrt{122}}.$$

7. Four distinct circles of radius r are on the surface of a unit sphere such that they are pairwise tangent. Find r.

Proposed by Thomas Lam

Answer. $\sqrt{\frac{2}{3}}$

Solution. The 6 tangency points by symmetry form an octahedron, An octahedron inscribed in a unit sphere has a side length of $1/\sqrt{2}$. Each circle has three of these tangency points on it in an equilateral triangle which is side length $1/\sqrt{2}$, so it must have radius $\boxed{\sqrt{2}/\sqrt{3}}$.

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8. Let ω be a unit circle with center O and diameter AB. A point C is chosen on ω. Let M, N be the midpoints of arc AC, BC, respectively, and let AN, BM intersect at I. Suppose that AM, BC, OI concur at a point. Find the area of $\triangle ABC$.

Proposed by Kevin You

Answer. $\frac{24}{25}$

Solution. Let the intersection of AM, BC, OI be P. Additionally, let D be the midpoint of BC.

Claim 1. $\triangle AIP$ is a 45-45-90 triangle.
Note that $\triangle ABP$ is an isosceles triangle, and since I is on altitude BM, $\triangle AIP$ is also isosceles. Additionally, since M, N are midpoints of their respective arcs, arc MN extends $90°$, so an inscribed angle, $\angle MAN = \angle PAI = 45°$. Henceforth, $\triangle AIP$ is a 45-45-90 triangle, as desired.

Claim 2. $\triangle CID$ is a 45-45-90 triangle. From the previous step, OI is perpendicular to AN, and additionally $\angle IAO = \angle IAC = \angle INO$, hence we have that $AI = IN$. So, I is equidistant between two parallel lines AC, ON, which implies $IC = ID$. Next, since $\angle AIP = \angle ACP = 90°$, we have $AICP$ is cyclic. Thus, $\angle DCI = 180° - \angle PCI == \angle PAI = 45°$, and $\triangle DCI$ is also a 45-45-90 triangle.

Claim 3. $\triangle AIC \cong \triangle CID$.
We already have $AI = PI$ and $CI = DI$. Moreover, $\angle AIC = \angle AIP + \angle PIC = 90° + \angle PIC = \angle CID + \angle PIC = \angle PID$. Hence, $\triangle AIC \cong \triangle CID$ (in fact, the two triangles are related by a $90°$ rotation about I).

Finally, we obtain from the triangle congruence that $AC = PD$. However, $AB = PB$, which means $AB = PB = PD + DB = AC + BC/2$. This gives us a relation between the three side lengths of $\triangle ABC$, and along with the fact that $\triangle ACB$ is right allows us to see that $\triangle ACB$ is a 3-4-5 triangle. The answer is $\boxed{\frac{24}{25}}$.

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9. Let $\triangle ABC$ be a triangle with circumcenter O satisfying $AB = 13$, $BC = 15$, and $AC = 14$. Suppose there is a point P such that $PB \perp BC$ and $PA \perp AB$. Let X be a point on AC such that $BX \perp OP$. What is the ratio AX/XC?

Proposed by Thomas Lam

Answer. $\frac{169}{225}$

Solution. I claim that BX is the symmedian of ABC. Reflect B over A to B' and draw circle ω, centered at P through B (and B'). Also draw circumcircle Ω of ABC. then line BX is the radical axis between ω and Ω.

Now invert about B. B'^* is the midpoint of BA^*, Ω maps to A^*C^*, and ω maps to a line parallel to BC but passing through B'^*. Thus, as BC is the same line as BC^*, ω maps to the line through B'^* and the median of A^*C^*. This means that the line BX, which is the same as the line through B and the intersection of ω and Ω, must be the median of the triangle A^*BC^*. Inverting back reflects this over the angle bisector, so BX must be a symmedian.

Thus, the final answer would be $13^2/15^3 = \boxed{\frac{169}{225}}$

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10. The vertices of $\triangle ABC$ are labeled in counter-clockwise order, and its sides have lengths $CA = 2022$, $AB = 2023$, and $BC = 2024$. Rotate B $90°$ counter-clockwise about A to get a point B'. Let D be the orthogonal projection of B' unto line AC, and let M be the midpoint of line segment BB'. Then ray BM intersects the circumcircle of $\triangle CDM$ at a point $N \neq M$. Compute MN.

Proposed by Thomas Lam

Answer. $2\sqrt{2}$

Solution. In general, if a, b, c denote the usual sides, and $a > b$, then $MN = \frac{a^2-b^2}{\sqrt{2}c}$. If $b, c, a = n, n+1, n+2$ for a positive integer n, this magically simplifies to $\boxed{2\sqrt{2}}$.

Begin by letting C' be the intersection of the A-altitude and line $B'D$. The crucial observation is that $\triangle ABC \cong \triangle B'AC'$. Moreover both these triangles are positively oriented, so there exists a rotation that sends one to the other. A moment of inspection reveals that the center of this rotation is M.

Label the feet of the A, B, and C-altitudes as E, F, G. By the rotation observation, we see that $\angle CMC' = 90°$, and so it follows that $CDNMEC'$ is a cyclic hexagon. By Power of a Point, we obtain

$$(B'M - MN) \cdot B'M = B'D \cdot B'C' = AF \cdot b$$

and

$$BM \cdot (BM + MN) = BE \cdot BC = BE \cdot a.$$

Subtracting, and using $BM = B'M$, we see that

$$BE \cdot a - AF \cdot b = (BM)(2MN) = (AB\sqrt{2})MN = c\sqrt{2}MN.$$

So $MN = \frac{BE \cdot a - AF \cdot b}{\sqrt{2}c}$. This is computable as-is, but notice that $BE \cdot a, AF \cdot b$ are the powers of A and B to the circumcircle of $\triangle CEF$, so by Power of a Point we have that $BE \cdot a - AF \cdot b = BP^2 - AP^2$, where P is the circumcenter of $\triangle CEF$. But P lies on the C-altitude, so by applying the Pythagorean Theorem we in fact have that $BP^2 - AP^2 - a^2 - b^2$. Thus we have proven that, indeed, $MN = \frac{a^2-b^2}{\sqrt{2}c}$

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11. The surface of a table is an ellipse with semimajor axis 4 and semiminor axis 2 (imagine starting with a circle of radius 1, stretching it by a factor of 4 in the x direction, and then stretching that by a factor of 2 in the y-direction). A circular coin of radius 1 is dropped uniformly randomly onto the table such that its center is on the table. Approximate the probability that the entire coin is on the table (i.e. that no part of the coin is hanging off the table). Express your answer as a decimal rounded to 6 places.

Proposed by Connor Gordon

Answer. 0.3540178