CMIMC 2023

Geometry Round

Instructions

1. Do not look at the test before the proctor starts the round.
2. This test consists of 10 short-answer problems to be solved in 60 minutes. The final estimation question will be used to break ties.
3. No computational aids other than pencil/pen are permitted.
4. Write your name and team name on your answer sheet.
5. Write your answers in the corresponding lines on your answer sheet.
6. Answers must be reasonably simplified.
7. If you believe that the test contains an error, submit your protest to the 2023 CMIMC discord.

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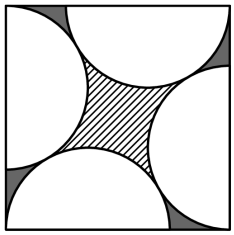
Geometry

1. Triangle ABC is isosceles with $AB = AC$. The bisectors of angles ABC and ACB meet at I. If the measure of angle CIA is 130°, compute the measure of angle CAB.

2. Two circles have radius 2 and 3, and the distance between their centers is 10. Let E be the intersection of their two common external tangents, and I be the intersection of their two common internal tangents. Compute EI.

   (A *common external tangent* is a tangent line to two circles such that the circles are on the same side of the line, while a *common internal tangent* is a tangent line to two circles such that the circles are on opposite sides of the line).

3. Four semicircles of radius 1 are placed in a square, as shown below. The diameters of these semicircles lie on the sides of the square and each semicircle touches a vertex of the square. Find the absolute difference between the shaded area and the "hatched" area.



4. A rhombus $\mathcal{R}$ has short diagonal of length 1 and long diagonal of length 2023. Let $\mathcal{R}'$ be the rotation of $\mathcal{R}$ by 90° about its center. If $\mathcal{U}$ is the set of all points contained in either $\mathcal{R}$ or $\mathcal{R}'$ (or both; this is known as the *union* of $\mathcal{R}$ and $\mathcal{R}'$) and $\mathcal{I}$ is the set of all points contained in both $\mathcal{R}$ and $\mathcal{R}'$ (this is known as the *intersection* of $\mathcal{R}$ and $\mathcal{R}'$, compute the ratio of the area of $\mathcal{I}$ to the area of $\mathcal{U}$.

5. In trapezoid $ABCD, AB = 3, BC = 2, CD = 5$, and $\angle B = \angle C = 90^\circ$. The angle bisectors of $\angle A$ and $\angle D$ intersect at a point P in the interior of $ABCD$. Compute $BP^2 + CP^2$.

6. Let $ABCD$ be a regular tetrahedron. Suppose points X, Y, and Z lie on rays AB, AC, and AD respectively such that $XY = YZ = 7$ and $XZ = 5$. Moreover, the lengths AX, AY, and AZ are all distinct. Find the volume of tetrahedron $AXYZ$.

7. Four distinct circles of radius r are on the surface of a unit sphere such that they are pairwise tangent. Find r.

8. Let ω be a unit circle with center O and diameter AB. A point C is chosen on ω. Let M, N be the midpoints of arc AC, BC, respectively, and let AN, BM intersect at I. Suppose that AM, BC, OI concur at a point. Find the area of $\triangle ABC$.

9. Let $\triangle ABC$ be a triangle with circumcenter O satisfying $AB = 13$, $BC = 15$, and $AC = 14$. Suppose there is a point P such that $PB \perp BC$ and $PA \perp AB$. Let X be a point on AC such that $BX \perp OP$. What is the ratio AX/XC?

10. The vertices of $\triangle ABC$ are labeled in counter-clockwise order, and its sides have lengths $CA = 2022$, $AB = 2023$, and $BC = 2024$. Rotate B 90° counter-clockwise about A to get a point B'. Let D be the orthogonal projection of B' unto line AC, and let M be the midpoint of line segment BB'. Then ray BM intersects the circumcircle of $\triangle CDM$ at a point $N \neq M$. Compute MN.

11. **(Tiebreaker)** *The surface of a table is an ellipse with semimajor axis* 4 *and semiminor axis* 2 *(imagine starting with a circle of radius* 1*, stretching it by a factor of* 4 *in the* x *direction, and then stretching that by a factor of* 2 *in the* y*-direction). A circular coin of radius* 1 *is dropped uniformly randomly onto the table such that its center is on the table. Approximate the probability that the entire coin is on the table (i.e. that no part of the coin is hanging off the table). Express your answer as a decimal rounded to* 6 *places.*