Combinatorics and Computer Science Round Solutions

1. Oh no! While playing Mario Party, Theo has landed inside the Bowser Zone. If his next roll is between 1 and 5 inclusive, Bowser will shoot his “Zero Flame” that sets a player’s coin and star counts to zero. Fortunately, Theo has a double dice block, which lets him roll two fair 10-sided dice labeled 1-10 and take the sum of the rolls as his “roll”. If he uses his double dice block, what is the probability he escapes the Bowser zone without losing his coins and stars?

   Proposed by Connor Gordon

   Answer. $\frac{9}{10}$

   Solution. There are $10^2 = 100$ possible outcomes of the two dice rolls. To get a sum of 5 or less, there are few enough options that we can list them out: 1-1, 1-2, 1-3, 1-4, 2-1, 2-2, 2-3, 3-1, 3-2, 4-1, for a total of 10 options (one can also count the number of solutions to $x + y \leq 5$ via counting the number of solutions to $x + y + z = 5$ with $x, y \geq 1$ and $z \geq 0$ which is $\binom{5}{2} = 10$ by stars and bars. In any case, there are $100 - 10 = 90$ successful outcomes out of 100 total, for a probability of $\frac{90}{100} = \boxed{\frac{9}{10}}$.

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2. Find the natural number A such that there are A integer solutions to $x + y \geq A$ where $0 \leq x \leq 6$ and $0 \leq y \leq 7$.

   Proposed by David Tang

   Answer. 10

   Solution. Observe that this is unique as the number of solutions increases as A decreases. Consider $A = 10$, the number of solutions is $4 + 3 + 2 + 1 = 10$ by casing on the possible values for y given $x = 6, 5, 4, 3$.

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3. Clarabelle wants to travel from $(0, 0)$ to $(6, 2)$ in the coordinate plane. She is able to move in one-unit steps up, down, or right, must stay between $y = 0$ and $y = 2$ (inclusive), and is not allowed to visit the same point twice. How many paths can she take?

   Proposed by Connor Gordon

   Answer. 729

   Solution. Observe that there is a one-to-one correspondence between $\{0, 1, 2\}$-valued sequences and such paths, where a value of j in the ith term in the sequence corresponds to (i, j) being the last point Clarabelle visits where $x = i$ (verify that indeed this is a one-to-one correspondence). With this in hand, the number of such sequences is clearly $3^6 = \boxed{729}$.

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4. Evaluate $1 \oplus 2 \oplus \cdots \oplus 987654321$ where $\oplus$ is bitwise exclusive OR.

   ($A \oplus B$ in binary has an n-th digit equal to 1 if the n-th binary digits of A and B differ and 0 otherwise. For example, $5 \oplus 9 = 0101_2 \oplus 1001_2 = 1100_2 = 12$ and $6 \oplus 7 = 110_2 + 111_2 = 001_2 = 1$.)

   Proposed by Jacob Weiner

   Answer. 1

   Solution. We claim that if $f(k) = 1 \oplus 2 \oplus \cdots \oplus n = \begin{cases} n, n \equiv 0 \mod 4 \\ 1, n \equiv 1 \mod 4 \\ n+1, n \equiv 2 \mod 4 \\ 0, n \equiv 3 \mod 4 \end{cases}$.

   Clearly true for $f(0) = 0$.

   If it's true for a particular n...

   If $n \equiv 0 \mod 4$: $f(n) = n$ so $f(n+1) = n \oplus (n+1) = 1$ because n is even.

   If $n \equiv 1 \mod 4$: $f(n) = 1$ so $f(n+1) = 1 \oplus (n+1) = n+2$ because $n+1$ is even.

   If $n \equiv 2 \mod 4$: $f(n) = n+1$ so $f(n+1) = (n+1) \oplus (n+1) = 0$.

   If $n \equiv 3 \mod 4$, $f(n) = 0$ so $f(n+1) = (n+1) \oplus 0 = n+1$.

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5. A BWM tree is defined recursively:

   - An empty tree is a BWM tree of height 0 and size 0.
   - A nonempty BWM tree consists of a root node and three subtrees, each of which is itself a (possibly empty) BWM tree. The height of the tallest of the subtrees must be at most 2 more than the height of the shortest.
   - The height of a nonempty BWM tree is one more than the height of its tallest subtree, and the size of a nonempty BWM tree is one more than the sum of the sizes of the subtrees.

   What is the minimum size of a height-10 BWM tree?

   Proposed by Jacob Weiner

   Answer. 154

   Solution. A height-0 BWM tree has at least 0 nodes.

   A height-1 BWM tree has at least 1 node.

   A height-2 BWM tree has at least 2 nodes.

   A height-$n+3$ BWM tree of minimal size has two subtrees of height n and one of height $n+2$. So $h(n+3) = h(n+2) + 2h(n) + 1$.

So:

$$\begin{aligned} h(3) &= 3 \\ h(4) &= 6 \\ h(5) &= 11 \\ h(6) &= 18 \\ h(7) &= 31 \\ h(8) &= 54 \\ h(9) &= 91 \\ h(10) &= 154 \end{aligned}$$

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6. Compute the number of five-digit positive integers whose digits have exactly 30 distinct permutations (the permutations do not necessarily have to be valid five-digit integers).

   Proposed by David Sun

   Answer. 9720

   Solution. First notice that $30 = \frac{5!}{4}$, therefore there must be exactly two digits repeated twice because 4 does not equal $k!$ for any positive integer k, but it is equal to $2! \cdot 2!$. Five-digit positive integers that have exactly 30 distinct permutations must have three distinct digits. There are $\binom{10}{3} = 120$ ways to choose a number's three distinct digits. There are 3 ways to choose which of those is unique—the remaining two are both repeated—and there are 5 choices for the unique digit's place. There are $\binom{4}{2} = 6$ ways to choose the places for the first repeated digit, which leaves only one way to choose the places for the remaining repeated digit. Therefore there are $120 \cdot 3 \cdot 5 \cdot 6 = 10800$ five-digit strings that have exactly 30 distinct permutations. However, we must remove the strings that begin with 0, because those do not correspond to any five-digit positive integer. We first consider the case in which 0 is both the leading digit and the unique digit. There are $\binom{9}{2} = 36$ choices for the other two distinct digits and $\binom{4}{2} = 6$ arrangements of those digits, so there are $36 \cdot 6 = 216$ such strings in this case. For the case in which 0 is one of the two repeated digits, there are 4 positions for the other 0, $\binom{9}{2} = 36$ choices for the remaining two distinct digits, 2 choices for which of those two is unique, and 3 places for that unique digit. This yields $4 \cdot 36 \cdot 2 \cdot 3 = 864$ more strings that begin with 0, for a total of $216 + 864 = 1080$ strings to exclude, thus we have $10800 - 1080 = \boxed{9720}$ five-digit positive integers that have exactly 30 distinct permutations.

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7. Max has a light bulb and a defective switch. The light bulb is initially off, and on the nth time the switch is flipped, the light bulb has a $\frac{1}{2(n+1)^2}$ chance of changing its state (i.e. on $\to$ off or off $\to$ on). If Max flips the switch 100 times, find the probability the light is on at the end.

   Proposed by Connor Gordon

   Answer. $\frac{25}{101}$

Solution. Note we need an odd number of toggles in total. Expressing this in the language of generating functions, we are interested in the sum of the coefficients of the odd degree terms in

$$\prod_{n=1}^{100}\left(\frac{1}{2(n+1)^2}x+\left(1-\frac{1}{2(n+1)^2}\right)\right)$$

To get the sum of all terms, we simply plug in 1 to get a sum of 1. We can then get the even coefficients using the usual roots of unity filter $\frac{f(1)+f(-1)}{2}$. We already know $f(1)=1$, while

$$f(-1)=\prod_{n=1}^{100}\left(1-\frac{1}{(n+1)^2}\right)=\frac{51}{101},$$

where the computation of this product is as in the previous solution.

Thus the sum of the coefficients of the even degree terms is $\frac{1}{2}(1+\frac{51}{101})=\frac{76}{101}$. The sum of the coefficients of the odd degree terms is then $1-\frac{76}{101}=\boxed{\frac{25}{101}}$.

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8. How many functions $f:\{1,2,3,4,5,6\}\to\{1,2,3,4,5,6\}$ have the property that $f(f(x))+f(x)+x$ is divisible by 3 for all $x\in\{1,2,3,4,5,6\}$?

   Proposed by Kyle Lee

   Answer. 360

   Solution. First, observe that we must either have $x\equiv f(x)\equiv f(f(x)) \bmod 3$ or they're all distinct modulo 3. It then follows that for each $x\in S=\{1,2,3,4,5,6\}$ falls into exactly one of four sets: $A: f(x)\equiv x+1 \bmod 3$ $B: f(x)\equiv x-1 \bmod 3$ $C: f(x)\equiv x+3 \bmod 6$ $D: f(x)=x$. Now, we make some preliminary observations. It is easy to see that $x\in A \implies f(x)\in A$ and $x\in B \implies f(x)\in B$. However, if $x\in C$, we can have either $f(x)\in C$ or $f(x)\in D$.

   Case 1: A and B are non-empty. It is easy to see that there are 2^3 such functions here since A and B must each have exactly 3 elements.

   Case 2: A and B are both empty. Then each $x\in S$ must belong to either C or D. It is easy to see that there are 2^6 such functions here.

   Case 3: A is non-empty, B is empty. (The other case is the exact same by symmetry). Case 3.1: A has 3 elements: There are $2^3=8$ such functions. Case 3.2: A has 4 elements: This implies one residue modulo 3 shows up twice in A, while the other two residues show up only once; hence, there are $3\cdot 2^2\cdot 2=24$ such functions. Case 3.3: A has 5 elements: There are 6 ways to choose which element is absent from A. Hence, there are $6\cdot 2^3=48$ such functions. Case 3.4: A has 6 elements: There are simply $2^6=64$ such functions.

   The grand total is $2^3+2^6+2(8+24+48+64)=\boxed{360}$.

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9. A grid is called k-special if in each cell is written a distinct integer such that the set of integers in the grid is precisely the set of positive divisors of k. A grid is called k-awesome if it is k-special and for each positive divisor m of k, there exists an m-special grid within this k-special grid

(within meaning you could draw a box in this grid to obtain the new grid). Find the sum of the 4 smallest integers k for which no k-awesome grid exists.

Proposed by Oliver Hayman

Answer. 6774

Solution. A k-awesome grid exists only when $k = 1, p_1^a, p_1^a p_2^b, p_1^a p_2^b p_3, p_1^a p_2^b p_3 p_4$ for positive integers a, b, distinct primes p_1, p_2, p_3, p_4. Note all prime factors p must be adjacent to 1 as the only p-special grid is two adjacent cells with 1 and p, so it's impossible for k to have more than 4 prime factors. Let's say for distinct primes p_1, p_2, p_3, p_1^2, p_2^2, p_3^2 all divide k and a k-awesome grid exists, This means in some row or column of the grid, $p_i, 1, p_j$ will appear in that order. Assume WLOG $p_1, 1, p_2$ appear in that order in some row. Note the only p_1^2-special and p_2^2-special grids are 3 by 1 or 1 by 3 grids, so we must have $p_1^2, p_1, 1, p_2, p_2^2$ appear in some row or column. However, note the only p_1p_2-special grids are 2 by 2, 1 by 4, and 4 by 1 grids containing the factors of p_1, p_2, and none of these can exist with $p_1^2, p_1, 1, p_2, p_2^2$ appearing in that order. This shows that no k-awesome grid exists.
This shows the given k values are the only possible ones. For constructions:

1 - trivial p_1^a - 1 by p grid with cells labelled $1, p, p^2, \cdots$ in that order $p_1^a p_2^b$ - a by b grid where a cell in row i and column j is labelled $p_1^{i-1} p_2^{j-1}$. $p_1^a p_2^b p_3$ - instead of a grid, consider lattice points. Label the origin 1, label $(n, 0)$ with p_1^n and $(0, m)$ with p_2^m for all possible m and n, and label $(-n.0)$ with $p_3 \cdot p_1^{n-1}$ for all possible n. To fill in the rest of the grid, label each applicable point (x, y) with the product of the labels of $(x, 0)$ and $(0, y)$. $p_1^a p_2^b p_3 p_4$ - similar to the above strategy, except also label $(0, -m) = p_4 p_2^{m-1}$ for suitable m.

The 4 smallest values of k for which no k-awesome grid exists are $900 = 2^2 3^2 5^2$, $1764 = 2^2 3^2 7^2$, $1800 = 2^3 \cdot 3^2 \cdot 5^2$, $2310 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$. Their sum is $\boxed{6774}$.

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10. Each of the positive integers from 1 to 2023, inclusive, are randomly colored either blue or red. For each nonempty subset of $S = \{1, 2, \cdots, 2023\}$, we define the score of that subset to be the positive difference between the number of blue integers and the number of red integers in that subset. Let X be the expected value of the sum of the scores of all the nonempty subsets of S. What is the maximum integer k such that 2^k divides $2^{2023} \cdot X$?

Proposed by Kyle Lee

Answer. 9

Solution. Replace 8 with general n. We claim that the expected value is simply

$$\frac{n\binom{2n}{n}}{2^n}.$$

We try to count the contribution of a single element to the sum, let's say 1, with a subset of size $i+1$. If it's red, and we choose the other i elements, the contribution over varying all colour combinations is $\binom{i}{\lfloor i/2 \rfloor}$ as:
If i is even, we will never get a tie, so i to $i/2$ reds gives $+1$ and $i/2 - 1$ to 0 reds gives -1. Choosing i reds is matched with choosing 0 reds, etc, giving only $\binom{i}{i/2}$ remaining unmatched +1s. If i is odd, we ignore the tie with $\lfloor i/2 \rfloor$ reds. The other cases all match similarly except $\lfloor i/2 \rfloor + 1$

but $\binom{i}{\lfloor i/2 \rfloor} = \binom{i}{\lfloor i/2 \rfloor + 1}$ as i is odd, so it still holds in this case.
We can have 2 ways to colour 1 and 2^{n-1-i} ways to colour the rest. Thus, we wish to prove that $\sum_{i=0}^{n-1} \binom{n-1}{i}\binom{i}{\lfloor i/2 \rfloor}2^{n-i} = \binom{2n}{n}$.
Rearranging with $k = n-1$ and using the fact that $\binom{2n}{n} = 2\binom{2n-1}{n-1}$, we get that we want to prove $\sum_{i=0}^{k} \binom{k}{i}\binom{i}{\lfloor i/2 \rfloor}2^{k-i} = \binom{2k+1}{k+1}$.

We can imagine this as coloring $n-1$ balls with RGBY such that $B - Y \in \{0, 1\}$ as i is the number of blue plus yellows, and the other colors have 2 choices.

Now, we can model this as the generating function $(1/x + x + 1 + 1)^k$ where $1/x$ is yellow, x is blue, and 1 is either red or green. Then, this becomes $(1/\sqrt{x} + \sqrt{x})^{2k}$ and we want the x^0 plus x^1 terms, as those are when the difference in blue and yellow is 0 or 1.

These terms are $\binom{2k}{k} + \binom{2k}{k+1} = \binom{2k+1}{k+1}$ as desired.

Thus, this concludes the proof.

Now, we want to find $v_2\binom{4046}{2023}$ as 2023 is odd. The numerator has a v_2 of 4037 while the denominator has a v_2 of $2014 * 2 = 4028$. Thus, the answer is just $4037 - 4028 = \boxed{9}$.

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11. Polly is walking randomly in 5-dimensional space. She starts at $(0, 0, 0, 0, 0)$, and every second she steps to one of the 32 points with integer coordinates that are a distance of 1 away from her current location. Approximate the probability that, after she first steps away from $(0, 0, 0, 0, 0)$, she ever returns to $(0, 0, 0, 0, 0)$. Express your answer as a decimal rounded to 6 places.

Proposed by Connor Gordon

Answer. 0.1351786