

CMIMC 2023

Combinatorics and Computer Science Round

Instructions

1. Do not look at the test before the proctor starts the round.
2. This test consists of 10 short-answer problems to be solved in 60 minutes. The final estimation question will be used to break ties.
3. No computational aids other than pencil/pen are permitted.
4. Write your name and team name on your answer sheet.
5. Write your answers in the corresponding lines on your answer sheet.
6. Answers must be reasonably simplified.
7. If you believe that the test contains an error, submit your protest to the 2023 CMIMC discord.



Combinatorics and Computer Science

1. Oh no! While playing Mario Party, Theo has landed inside the Bowser Zone. If his next roll is between 1 and 5 inclusive, Bowser will shoot his “Zero Flame” that sets a player’s coin and star counts to zero. Fortunately, Theo has a double dice block, which lets him roll two fair 10-sided dice labeled 1-10 and take the sum of the rolls as his “roll”. If he uses his double dice block, what is the probability he escapes the Bowser zone without losing his coins and stars?
2. Find the natural number A such that there are A integer solutions to $x + y \geq A$ where $0 \leq x \leq 6$ and $0 \leq y \leq 7$.
3. Clarabelle wants to travel from $(0, 0)$ to $(6, 2)$ in the coordinate plane. She is able to move in one-unit steps up, down, or right, must stay between $y = 0$ and $y = 2$ (inclusive), and is not allowed to visit the same point twice. How many paths can she take?
4. Evaluate $1 \oplus 2 \oplus \dots \oplus 987654321$ where $\oplus$ is bitwise exclusive OR.

($A \oplus B$ in binary has an n -th digit equal to 1 if the n -th binary digits of A and B differ and 0 otherwise. For example, $5 \oplus 9 = 0101_2 \oplus 1001_2 = 1100_2 = 12$ and $6 \oplus 7 = 110_2 \oplus 111_2 = 001_2 = 1$.)

5. A BWM tree is defined recursively:
 - An empty tree is a BWM tree of height 0 and size 0.
 - A nonempty BWM tree consists of a root node and three subtrees, each of which is itself a (possibly empty) BWM tree. The height of the tallest of the subtrees must be at most 2 more than the height of the shortest.
 - The height of a nonempty BWM tree is one more than the height of its tallest subtree, and the size of a nonempty BWM tree is one more than the sum of the sizes of the subtrees.

What is the minimum size of a height-10 BWM tree?

6. Compute the number of five-digit positive integers whose digits have exactly 30 distinct permutations (the permutations do not necessarily have to be valid five-digit integers).
7. Max has a light bulb and a defective switch. The light bulb is initially off, and on the n th time the switch is flipped, the light bulb has a $\frac{1}{2(n+1)^2}$ chance of changing its state (i.e. on $\rightarrow$ off or off $\rightarrow$ on). If Max flips the switch 100 times, find the probability the light is on at the end.
8. How many functions $f : \{1, 2, 3, 4, 5, 6\} \rightarrow \{1, 2, 3, 4, 5, 6\}$ have the property that $f(f(x)) + f(x) + x$ is divisible by 3 for all $x \in \{1, 2, 3, 4, 5, 6\}$?
9. A grid is called k -special if in each cell is written a distinct integer such that the set of integers in the grid is precisely the set of positive divisors of k . A grid is called k -awesome if it is k -special and for each positive divisor m of k , there exists an m -special grid within this k -special grid (within meaning you could draw a box in this grid to obtain the new grid). Find the sum of the 4 smallest integers k for which no k -awesome grid exists.
10. Each of the positive integers from 1 to 2023, inclusive, are randomly colored either blue or red. For each nonempty subset of $S = \{1, 2, \dots, 2023\}$, we define the score of that subset to be the positive difference between the number of blue integers and the number of red integers in that subset. Let X be the expected value of the sum of the scores of all the nonempty subsets of S . What is the maximum integer k such that 2^k divides $2^{2023} \cdot X$?
11. **(Tiebreaker)** Polly is walking randomly in 5-dimensional space. She starts at $(0, 0, 0, 0, 0)$, and every second she steps to one of the 32 points with integer coordinates that are a distance of 1 away from her current location. Approximate the probability that, after she first steps away from $(0, 0, 0, 0, 0)$, she ever returns to $(0, 0, 0, 0, 0)$. Express your answer as a decimal rounded to 6 places.