Algebra and Number Theory Round Solutions

1. Suppose a, b, c, and d are non-negative integers such that

$$(a+b+c+d)(a^2+b^2+c^2+d^2)^2 = 2023.$$

Find $a^3+b^3+c^3+d^3$.

Proposed by Connor Gordon

Answer. 43

Solution. Let $M = a+b+c+d$ and $N = a^2+b^2+c^2+d^2$; we have $MN^2 = 2023$. Factoring $2023 = 7 \cdot 17^2$, we have $(M,N) \in \{(2023,1),(7,17)\}$. The former is clearly absurd, so $M = 7$ and $N = 17$.

We now look for ways to write 17 as a sum of four squares. WLOG $a \leq b \leq c \leq d$. Clearly $3 \leq d \leq 4$.

If $d = 4$, we must have $1 = a^2+b^2+c^2$ which only admits $(a,b,c) = (0,0,1)$, but $0+0+1+4 \neq 7$, so this case fails.

If $d = 3$, we must have $8 = a^2+b^2+c^2$. It's easy to see we must have $(a,b,c) = (0,2,2)$, and indeed $0+2+2+3 = \boxed{7}$.

Thus $(a,b,c,d) = (0,2,2,3)$, and thus $a^3+b^3+c^3+d^3 = \boxed{43}$.

Remark: Rearranging this gives the rather pretty

$$(2+0+2+3)(2^2+0^2+2^2+3^2) = 2023.$$

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2. Find the largest possible value of a such that there exist real numbers $b, c > 1$ such that

$$a^{\log_b c} \cdot b^{\log_c a} = 2023.$$

Proposed by Howard Halim

Answer. $\sqrt{2023}$

Solution. Convert everything to base e, and let $A = \ln a, B = \ln b, C = \ln c$. Since $a^{\log_b c} = e^{\frac{AC}{B}}$ and $b^{\log_c a} = e^{\frac{BA}{C}}$, we have

$$A\left(\frac{B}{C}+\frac{C}{B}\right) = \ln(2023)$$

A is maximized when $\frac{B}{C} = \frac{C}{B} = 1$ by AM-GM. Its maximum value is $A = \frac{\ln(2023)}{2}$, so the maximum value of a is $\boxed{\sqrt{2023}}$.

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3. Compute

$$2022^{\left(2022^{\cdot^{\cdot^{\left(2022^{2022}\right)}}}\right)} \pmod{111}$$

where there are 2022 2022s. (Give the answer as an integer from 0 to 110).

Proposed by David Tang

Answer. 75

Solution. We notice that the exponent power tower is some multiple of $2022 * 2022$ which is a multiple of 36. Thus, we know that the power is 0 mod 36 so the final answer is $2022^0 = 1$ mod 37. Similarly, we know that 2022 is 0 mod 3 so by CRT, we have the answer is $\boxed{75}$ mod 111.

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4. An arithmetic sequence of exactly 10 positive integers has the property that any two elements are relatively prime. Compute the smallest possible sum of the 10 numbers.

Proposed by Kyle Lee

Answer. 1360

Solution. We claim that the answer is $\boxed{1360}$, achievable with the sequence $1, 1+30, 1+30(2), \cdots, 1+30(9)$. Let a be the first term of the sequence and d be the common difference so that the sequence is $a, a+d, a+2d, \cdots, a+9d$. The sum of the integers is $10a+45d$. Clearly, $a \geq 1$.

If $2 \nmid d$, then at least one of $a, a+d$ is even and at least one of $a+2d, a+3d$ is even, regardless of what a is. But then that means two terms in the sequence both have a factor of 2, contradiction. Hence, $2 \mid d$.

Similarly, if $3 \nmid d$, at least one of $a, a+d, a+2d$ is divisible by 3 and at least one of $a+3d, a+4d, a+5d$ is divisible by 3, contradiction. Hence, $3 \mid d$.

Similarly, $5 \mid d$. Thus, $30 \mid d$, implying $d \geq 30$.

Thus, the smallest possible sum is $10a + 45d \geq 10 \cdot 1 + 45 \cdot 30 = \boxed{1360}$, as desired.

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5. Let $\mathcal{P}$ be a parabola that passes through the points $(0,0)$ and $(12,5)$. Suppose that the directrix of $\mathcal{P}$ takes the form $y = b$. (Recall that a parabola is the set of points equidistant from a point called the focus and line called the directrix) Find the minimum possible value of $|b|$.

Proposed by Kevin You

Answer. 4

Solution. The answer is 4. $\mathcal{P}$ has a focus f somewhere, such that the distance from the focus to the two points is equal to the distance from the two points to the directrix, respectively. This common value is equal to $|b-0| + |b-5|$.

However, by the triangle inequality on $(0,0),(12,5)$ and f, we must have

$$|b-0|+|b-5| = \|f-(0,0)\| + \|f-(12,5)\| \geq \|(12,5)-(0,0)\| = 13.$$

It follows that $b=-4$ is the value to minimize $|b|$. In this case, the focus lies on the line joining the two points.

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6. Compute the sum of all positive integers N for which there exists a unique ordered triple of non-negative integers (a,b,c) such that $2a+3b+5c=200$ and $a+b+c=N$.

   Proposed by Kyle Lee

   Answer. 280

   Solution. Eliminating the variable a, this is equivalent to there being a unique pair of non-negative integers such that $b+3c=200-2N$ and $b+c\leq N$.

   With $b+c\leq N$, we have the bounds $0\leq b+3c\leq 3N$, which implies $0\leq 200-2N\leq 3N$ or $40\leq N\leq 100$.

   Let $N=3k$ for some integer k. Then, both the pairs $(2,66-2k)$ and $(5,65-2k)$ are valid for $14\leq k\leq 32$, which means there's no unique solution for these N.

   Similarly, if $N=3k+1$, both pairs $(0,66-2k)$ and $(3,65-2k)$ work.

   Similarly, if $N=3k+2$, both pairs $(1,65-2k)$ and $(4,64-2k)$ work.

   This eliminates all N in range besides $40, 41, 99$, and 100. A quick check shows these all have unique solutions, so the answer is $40+41+99+100=\boxed{280}$.

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7. Let $\phi(n)$ denote the number of positive integers less than or equal to n which are relatively prime to n. Compute $\displaystyle\sum_{i=1}^{\phi(2023)} \frac{\gcd(i,\phi(2023))}{\phi(2023)}$.

   Proposed by Giacomo Rizzo

   Answer. $\frac{385}{34}$

   Solution. Note that $\gcd(x,\phi(2023))=d$ if and only if $\dfrac{x}{d}$ and $\dfrac{\phi(2023)}{d}$ are relatively prime. It follows that there are $\phi\left(\dfrac{\phi(2023)}{d}\right)$ values under $\phi(2023)$ for x that satisfy $\gcd(x,\phi(2023))=d$, so

$$\begin{aligned}
\sum_{i=1}^{\phi(2023)} \frac{\gcd(i,\phi(2023))}{\phi(2023)} &= \sum_{d\mid\phi(2023)} \frac{d\cdot\phi\left(\frac{\phi(2023)}{d}\right)}{\phi(2023)} \\
&= \sum_{d\mid\phi(2023)} \frac{\frac{\phi(2023)}{d}\cdot\phi(d)}{\phi(2023)} \\
&= \sum_{d\mid\phi(2023)} \frac{\phi(d)}{d}.
\end{aligned}$$

Note that $\phi(2023) = 2023(1-\frac{1}{7})(1-\frac{1}{17}) = 2^5 \cdot 3 \cdot 17$, so each $d \mid \phi(2023)$ can be written uniquely as abc, where $a \mid 2^5$, $b \mid 3$, and $c \mid 17$. Then, since ϕ is multiplicative,

$$\begin{aligned}\sum_{d\mid\phi(2023)} \frac{\phi(d)}{d} &= \sum_{a\mid 2^5} \frac{\phi(a)}{a} \cdot \sum_{b\mid 3} \frac{\phi(b)}{b} \cdot \sum_{c\mid 17} \frac{\phi(c)}{c} \\ &= \left(1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right)\left(1+\frac{2}{3}\right)\left(1+\frac{16}{17}\right) \\ &= \boxed{\frac{385}{34}}.\end{aligned}$$

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8. Consider digits $\underline{A}, \underline{B}, \underline{C}, \underline{D}$, with $\underline{A} \neq 0$, such that $\underline{ABCD} = (\underline{CD})^2 - (\underline{AB})^2$. Compute the sum of all distinct possible values of $\underline{A} + \underline{B} + \underline{C} + \underline{D}$.

Proposed by Kyle Lee

Answer. 21

Solution. Let $\underline{AB} = i$ and $\underline{CD} = j$; clearly, $i, j \in \{10, 11, \cdots, 99\}$ and $i < j$. The given condition is equivalent to $j^2 - i^2 = 100i + j$. This equation be rewritten as $(j+i)(j-i) = j+i+99i$, which simplifies to $j - i = 1 + \dfrac{99i}{j+i}$. In particular, $\dfrac{99i}{j+i}$ must be an integer. Since i and j are two-digit positive integers, we can conclude that $\dfrac{i}{j+i} = \dfrac{M}{99}$ for some positive integer $10 \leq M \leq 99$. Hence, $j - i = 1 + M$ and $99i = M(i+j)$, which simplifies to $j = (\dfrac{99}{M} - 1)i$. Then, $(\dfrac{99}{M} - 1 - 1)i = 1 + M$, which implies $i = \dfrac{(1+M)M}{99-2M}$.

From here, we require $99 - 2M \mid M^2 + M$, so $99 - 2M \mid 2M^2 + 2M$. Then,

$$99 - 2M \mid 2M^2 + 2M + M(99 - 2M) = 101M.$$

In particular, $99 - 2M \mid 202M$, so

$$99 - 2M \mid 202M + 101(99 - 2M) = 101 \cdot 99.$$

Now, since 101 is prime, we can easily see that $99 - 2M \mid 99$. However, if $99 - 2M \leq 11$, then $i \geq \frac{(56)55}{11} > 99$. Also, clearly $99 - 2M \neq 99$, so $99 - 2M = 33$. Thus, $M = 33, i = \frac{(34)33}{33} = 34$, and $j = 34 + 1 + 33 = 68$. Moreover, indeed $3468 = 68^2 - 34^2$, so we did not create any extraneous solutions and $3 + 4 + 6 + 8 = \boxed{21}$.

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9. Let n be a nonnegative integer less than 2023 such that $2n^2 + 3n$ is a perfect square. What is the sum of all possible n?

Proposed by Giacomo Rizzo

Answer. 444

Solution. We have $2\theta^2 + 3\theta = \theta(2\theta + 3)$. If $\gcd(\theta, 2\theta + 3) = 1$, then θ and $2\theta + 3$ are both perfect squares. Writing $x^2 = \theta$ and $y^2 = 2\theta + 3$, we see that $y^2 - 2x^2 = 3$. Perfect squares are equivalent to 0 or 1 modulo 3, so

$$\begin{aligned} y^2 &= 2x^2 + 3 \\ \Rightarrow x^2 &\equiv 0 \pmod 3 \\ \Rightarrow x^2 &\equiv 0 \pmod 9 \\ \Rightarrow y^2 &\equiv 3 \pmod 9, \end{aligned}$$

but this is a contradiction. Thus, $\gcd(\theta, 2\theta + 3) \neq 1$.

By the Euclidean Algorithm, $\gcd(\theta, 2\theta+3) = \gcd(\theta, 3)$, and since it doesn't equal 1, we just need to consider when it equals 3. Substituting $\theta = 3n$ into the expression gives $\theta(2\theta+3) = 9n(2n+1)$, so $n(2n + 1)$ is a perfect square. Also, since $\gcd(n, 2n + 1) = \gcd(n, 1) = 1$, we see that n and $2n+1$ are both perfect squares. Writing $x^2 = n$ and $y^2 = 2n+1$, we see that $y^2 - 2x^2 = 1$. Now, this is a Pell equation with solutions of the form $(3 + 2\sqrt{2})^k = y_k + x_k\sqrt{2}$ (with $k \in \mathbb{N}$) so we get $(x, y) = (0, 1), (2, 3), (12, 17), (70, 99), \ldots,$ and so on. Since $\theta = 3n = 3x^2$, it is less than 2023 only at $x = 0, 2, 12$. It follows that the sum of all possible θ is $0 + 12 + 432 = \boxed{444}$.

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10. For a given n, consider the points $(x, y) \in \mathbb{N}^2$ such that $x \leq y \leq n$. An ant starts from $(0, 1)$ and, every move, it goes from (a, b) to point (c, d) if $bc - ad = 1$ and d is maximized over all such points. Let g_n be the number of moves made by the ant until no more moves can be made. Find $g_{2023} - g_{2022}$.

    Proposed by David Tang

    Answer. 1632

    Solution. Only points which $\gcd(x, y) = 1$ get played on by the $bc - ad = 1$ condition. I claim that all of these points get played on.

    Consider the points by their slope to the origin, so $(x, y) \in \mathbb{Z}^2 \to x/y \in \mathbb{Q}$. This mapping is injective as $\gcd(x, y) = 1$. These fractions are all fractions in $[0, 1]$ such that the denominator is $\leq n$ since $0 \leq x \leq y \leq n$. Order these fractions in increasing order.

    Claim: Adjacent terms $a/b < c/d$ satisfy $bc - ad = 1$
    Proof: We induct on n, $n = 1$ is obvious.
    Consider the addition of a term in between $a/b < c/d$, and let this term be e/f where $f > b, f > d$. We know that $bc - ad = 1$ by induction and we can let $eb - af = r, cf - ed = s \in \mathbb{Z}^+$. Solving these three equations yields that $e = sa + rc, f = sb + rd$.

    We also notice that this is the first time we added a term between a/b and c/d as they used to be adjacent, so $f = sb + rd$ is minimized. which happens if $s = r = 1$ so the claim is true.

    Now, any move from (a, b) to (c, d) with $bc - ad = 1$ increases the slope by $1/(bd)$. Maximizing d means minimizing this difference in slope, which must be the next fraction (when viewed in increasing order) by the claim above.

Thus, all such points are hit as all fractions are hit. So g_n is $1 + \sum_{i=2}^{n}(\phi(i))$. The value we want is $\phi(2023) = \boxed{1632}$.

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11. Consider the sequence given by $x_1 = 1$ and $x_{n+1} = 1 + \frac{1}{x_n}$ for $n \geq 1$. As n grows large, x_n gets closer and closer to $\varphi = \frac{1+\sqrt{5}}{2}$. Approximate $\log_{1/2}|x_{2023} - \varphi|$. Express your answer as a non-negative integer.

    Proposed by Connor Gordon

    Answer. 2807.74