CMIMC 2023

Algebra and Number Theory Round

Instructions

1. Do not look at the test before the proctor starts the round.
2. This test consists of 10 short-answer problems to be solved in 60 minutes. The final estimation question will be used to break ties.
3. No computational aids other than pencil/pen are permitted.
4. Write your name and team name on your answer sheet.
5. Write your answers in the corresponding lines on your answer sheet.
6. Answers must be reasonably simplified.
7. If you believe that the test contains an error, submit your protest to the 2023 CMIMC discord.

Algebra and Number Theory

1. Suppose a, b, c, and d are non-negative integers such that
$$(a+b+c+d)(a^2+b^2+c^2+d^2)^2 = 2023.$$
Find $a^3+b^3+c^3+d^3$.

2. Find the largest possible value of a such that there exist real numbers $b, c > 1$ such that
$$a^{\log_b c} \cdot b^{\log_c a} = 2023.$$

3. Compute
$$2022^{\left(2022^{\cdot^{\cdot^{\left(2022^{2022}\right)}}}\right)} \pmod{111}$$
where there are 2022 2022s. (Give the answer as an integer from 0 to 110).

4. An arithmetic sequence of exactly 10 positive integers has the property that any two elements are relatively prime. Compute the smallest possible sum of the 10 numbers.

5. Let $\mathcal{P}$ be a parabola that passes through the points $(0,0)$ and $(12,5)$. Suppose that the directrix of $\mathcal{P}$ takes the form $y = b$. (Recall that a parabola is the set of points equidistant from a point called the focus and line called the directrix) Find the minimum possible value of $|b|$.

6. Compute the sum of all positive integers N for which there exists a unique ordered triple of non-negative integers (a,b,c) such that $2a+3b+5c = 200$ and $a+b+c = N$.

7. Let $\phi(n)$ denote the number of positive integers less than or equal to n which are relatively prime to n. Compute $\displaystyle\sum_{i=1}^{\phi(2023)} \frac{\gcd(i, \phi(2023))}{\phi(2023)}$.

8. Consider digits $\underline{A}, \underline{B}, \underline{C}, \underline{D}$, with $\underline{A} \neq 0$, such that $\underline{ABCD} = (\underline{CD})^2 - (\underline{AB})^2$. Compute the sum of all distinct possible values of $\underline{A} + \underline{B} + \underline{C} + \underline{D}$.

9. Let n be a nonnegative integer less than 2023 such that $2n^2+3n$ is a perfect square. What is the sum of all possible n?

10. For a given n, consider the points $(x,y) \in \mathbb{N}^2$ such that $x \leq y \leq n$. An ant starts from $(0,1)$ and, every move, it goes from (a,b) to point (c,d) if $bc - ad = 1$ and d is maximized over all such points. Let g_n be the number of moves made by the ant until no more moves can be made. Find $g_{2023} - g_{2022}$.

11. **(Tiebreaker)** *Consider the sequence given by* $x_1 = 1$ *and* $x_{n+1} = 1 + \frac{1}{x_n}$ *for* $n \geq 1$. *As* n *grows large,* x_n *gets closer and closer to* $\varphi = \frac{1+\sqrt{5}}{2}$. *Approximate* $\log_{1/2}|x_{2023} - \varphi|$. *Express your answer as a non-negative integer.*