CMIMC 2022
Team Round

INSTRUCTIONS

1. Do not look at the test before the proctor starts the round.
2. This test consists of 15 short-answer problems to be solved in 60 minutes.
3. Write your team name and team ID on your answer sheet.
4. Write your answers in the corresponding boxes on the answer sheets.
5. No computational aids other than pencil/pen are permitted.
6. If you believe that the test contains an error, submit your protest in writing to Doherty Hall 2302.

Team

1. Let $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$ be two squares such that the boundaries of $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$ does not contain any line segment. Construct 16 line segments A_iB_j for each possible $i, j \in \{1, 2, 3, 4\}$. What is the maximum number of line segments that don't intersect the edges of $A_1A_2A_3A_4$ or $B_1B_2B_3B_4$? (intersection with a vertex is not counted).

   Proposed by Allen Zheng

   Answer: 12

   Solution. We claim that the maximum is $\boxed{12}$; take 2 concentric squares with parallel edges for construction. To see we cannot do better, if there are 13 such edges then there exists one of $A_1, \ldots, A_4$, say A_1, such that all of $A_1B_i (1 \le i \le 4)$ are joined by edges that do not intersect the sides of $B_1B_2B_3B_4$. This means that A_1 must be in the interior of $B_1B_2B_3B_4$, and similarly one of the vertices of the square $B_1B_2B_3B_4$ must be in $A_1A_2A_3A_4$, say B_1. Then at most 2 edges not touching the sides of the squares can be drawn from each of A_2, A_4 giving a total of $\le 4{+}2{+}2{+}4 = 12$ edges, contradiction.

   Comment: During the contest we accepted 16 as an answer, since if one considers "intersection with vertex" geometrically then by placing one concentric square inside the other with parallel sides, the edges that do intersect the boundary of the inner square do so at their interior but their intersection is only with vertices of the inner square.

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2. Find the smallest positive integer n for which $315^2 - n^2$ evenly divides $315^3 - n^3$.

   Proposed by Kyle Lee

   Answer: 90

   Solution. Consider when $n \le 314$. We require

$$\frac{315^3 - n^3}{315^2 - n^2} = \frac{(315-n)(315^2 + 315n + n^2)}{(315-n)(315+n)} = \frac{315^2 + 315n + n^2}{315+n} = n + \frac{315^2}{n+315}$$

   to be an integer, so $n + 315 \mid 315^2$, where $n + 315 \in (315, 315 \cdot 2)$. Since $315^2 = 3^4 \cdot 5^2 \cdot 7^2$, we see that the possible values of $n + 315$ are $405, 441, 525, 567$, so the smallest possible value of n is $405 - 315 = \boxed{90}$.

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3. Let $ABCD$ be a rectangle with $AB = 10$ and $AD = 5$. Suppose points P and Q are on segments CD and BC, respectively, such that the following conditions hold:

   - $BD \parallel PQ$
   - $\angle APQ = 90^\circ$.

   What is the area of $\triangle CPQ$?

   Proposed by Kyle Lee

   Answer: $\frac{225}{16}$

Solution. Let $A = (0,5), B = (10,5), C = (10,0), D = (0,0)$ on the coordinate plane. Then $P = (a,0)$ and $Q = (10,b)$ for some reals $0 \le a \le 10$ and $0 \le b \le 5$. The first condition gives $b/(10-a) = 1/2$ and the second condition gives $-5/a = (a-10)/b$. Solving these two equations simultaneously gives $(a,b) = (5/2, 15/4)$. Then, the area of CPQ is simply $\frac{1}{2}b(10-a) = b^2 = \boxed{\dfrac{225}{16}}$.

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4. Let $\triangle ABC$ be equilateral with integer side length. Point X lies on $\overline{BC}$ strictly between B and C such that $BX < CX$. Let C' denote the reflection of C over the midpoint of $\overline{AX}$. If $BC' = 30$, find the sum of all possible side lengths of $\triangle ABC$.

   Proposed by Connor Gordon

   Answer: 130

   Solution. Note that $CAC'X$ is a parallelogram, so $\overline{BX}||\overline{AC'}$ and thus $XAC'B$ is a trapezoid. Let $\overline{AB}$ and $\overline{XC'}$ intersect at P. A simple angle chase shows that $\triangle APC'$ and $\triangle XPB$ are equilateral, so it follows that $AXBC'$ is an isosceles trapezoid and $AX = BC' = 30$.

   To finish, note that the length ℓ of an internal cevian of an equilateral triangle with side length s satisfies $\frac{s\sqrt{3}}{2} \le \ell < s$, so it follows that $30 < s \le 30 \cdot \frac{2}{\sqrt{3}} = \sqrt{1200}$, where one can easily note that $34 < \sqrt{1200} < 35$. It follows that the possible values for s are 31, 32, 33, and 34, for a sum of $\boxed{130}$.

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5. For any integer a, let $f(a) = |a^4 - 36a^2 + 96a - 64|$. What is the sum of all values of $f(a)$ that are prime?

   Proposed by Alexander Wang

   Answer: 22

   Solution. $(a^2)^2 - (6a-8)^2 = (a^2 - 6a + 8)(a^2 + 6a - 8) = (a-2)(a-4)(a^2 + 6a - 8)$. Two multiplicands must be either 1 or -1, so a can only be 1, 3, or 5, and 1 and 3 give values of 3 and 19 respectively, which sum to $\boxed{22}$.

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6. There are 9 points arranged in a 3×3 square grid. Let two points be adjacent if the distance between them is half the side length of the grid. (There should be 12 pairs of adjacent points). Suppose that we wanted to connect 8 pairs of adjacent points, such that all points are connected to each other. In how many ways is this possible?

   Proposed by Kevin You

   Answer: 192

   Solution. The key claim to simplify calculations is that there must be no cycles in the figure.

   If we look at the graph formed by the points and connected edges, we have a tree. This is because there are 9 points, and we only have enough edges to connect the points together; we do not have any edges to waste make cycles.

   So, we will consider casework, where we remove 4 edges out of 12 possible edges, so that all cycles are broken.

Let the edges be annotated as follows. We also assign numbers to outer edges.

$$\begin{bmatrix} & - & & - & \\ | & & | & & | \\ & - & & - & \\ | & & | & & | \\ & - & & - & \end{bmatrix} \quad \begin{bmatrix} & 1 & & 2 & \\ 8 & & | & & 3 \\ & - & & - & \\ 7 & & | & & 4 \\ & 6 & & 5 & \end{bmatrix}$$

We iterate through number of inner edges broken. Note that we have four kinds of cycles: the outer square (containing 4 squares), four L-shaped blocks (containing 3 squares), four rectangles (containing 2 squares), and 4 small squares (containing 1 square)

Case 1. All inner edges are broken.

The outer square is a cycle, and it fails.

Case 2. Three inner edges are broken.

$$\begin{bmatrix} & - & & - & \\ | & & \times & & | \\ & \times & & \times & \\ | & & | & & | \\ & - & & - & \end{bmatrix}$$

Note that all smaller cycles are broken; we just need to break the outer square. So, there are 4 symmetries for the inner edges, and 8 cases for the outside, yielding $4 \cdot 8$ cases.

Case 3a. Two inner edges are broken.

$$\begin{bmatrix} & - & & - & \\ | & & \times & & | \\ & \times & & - & \\ | & & | & & | \\ & - & & - & \end{bmatrix}$$

Here, all cycles are broken, except the L-shape block on the top left, and the small square on the bottom right. It follows that we need to break one edge from each of $\{4, 5\}$ and $\{1, 2, 3, 6, 7, 8\}$. We have $4 \cdot 2 \cdot 6$ cases.

Case 3b. Two inner edges are broken.

$$\begin{bmatrix} & - & & - & \\ | & & | & & | \\ & \times & & \times & \\ | & & | & & | \\ & - & & - & \end{bmatrix}$$

Here, all cycles are broken, except the two vertical rectangles. It follows that we need to break one edge from each of $\{2, 3, 4, 5\}$ and $\{1, 6, 7, 8\}$. We have $2 \cdot 4^2$ cases.

Case 4. One inner edges are broken.

$$\begin{bmatrix} & - & & - & \\ | & & \times & & | \\ & - & & - & \\ | & & | & & | \\ & - & & - & \end{bmatrix}$$

Here, we need to break the two bottom squares, and the upper rectangle. (The outer square and the L-shape will also be broken)

It follows that we need to break one edge from each of $\{4, 5\}$, $\{6, 7\}$, and $\{1, 2, 3, 8\}$. We have $4 \cdot 2^2 \cdot 4$ cases.

Case 5. Zero inner edges are broken.

$$\begin{bmatrix} & - & & - & \\ | & & | & & | \\ & - & & - & \\ | & & | & & | \\ & - & & - & \end{bmatrix}$$

Here, we need to break each small square. (All other cycles will also be broken)

It follows that we need to break one edge from each of $\{2, 3\}$, $\{4, 5\}$, $\{6, 7\}$, $\{1, 8\}$. We have 2^4 cases.

The total number of cases is

$$4 \cdot 8 + 4 \cdot 2 \cdot 6 + 2 \cdot 4^2 + 4 \cdot 2^2 \cdot 4 + 2^4 = \boxed{192}$$

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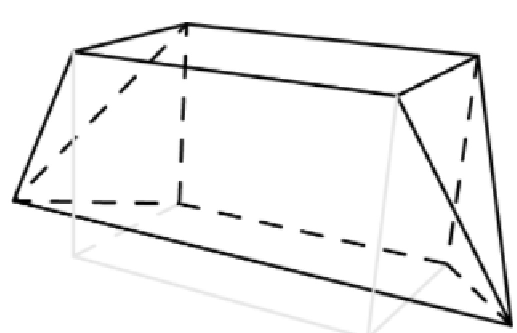
7. A $3 \times 2 \times 2$ right rectangular prism has one of its edges with length 3 replaced with an edge of length 5 parallel to the original edge. The other 11 edges remain the same length, and the 6 vertices that are not endpoints of the replaced edge remain in place. The resulting convex solid has 8 faces, as shown below.

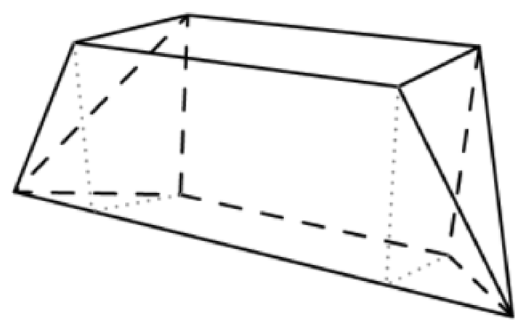


Find the volume of the solid.

Proposed by Justin Hsieh

Answer: $\dfrac{22+11\sqrt{2}}{3}$

Solution.



First, the new solid is still symmetric about the plane containing the midpoints of the edges parallel to the new edge. This is because the remaining edges must remain the same length, and the new edge is parallel to the old edge. Now consider coordinates: suppose the original prism had opposite vertices $(0,0,0)$ and $(3,2,2)$, and we replaced the edge from $(0,0,0)$ to $(3,0,0)$. The new edge has endpoints $(-1,y,z)$ and $(4,y,z)$ for some y,z. The point $(-1,y,z)$ is distance 2 from both $(0,2,0)$ and $(0,0,2)$. This allows us to conclude that $y=z=1-\sqrt{\frac{1}{2}}$.

Split the solid into two pyramids with base $2+\sqrt{2}$ and height 1, and a prism with base $2+\sqrt{2}$ and height 3, for a volume of $\boxed{\dfrac{22+11\sqrt{2}}{3}}$.

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8. There are 36 contestants in the CMU Puyo-Puyo Tournament, each with distinct skill levels.

The tournament works as follows: First, all $\binom{36}{2}$ pairings of players are written down on slips of paper and are placed in a hat. Next, a slip of paper is drawn from the hat, and those two players play a match. It is guaranteed that the player with a higher skill level will always win the match. We continue drawing slips (without replacement) and playing matches until the results of the match completely determine the order of skill levels of all 36 contestants (i.e. there is only one possible ordering of skill levels consistent with the match results), at which point the tournament immediately finishes. What is the expected value of the number of matches played before the stopping point is reached?

Answer: $\dfrac{22085}{36}$

Proposed by Dilhan Salgado

Solution. WLOG say the skill levels of the contestants are 1 through 36.

We claim that the tournament finishing is equivalent to the matches $(1,2),(2,3),(3,4),\ldots,(35,36)$ being played. First note that if all of these matches are played then we clearly will have a valid ranking order. Now, note that if any of these $(i,i+1)$ matches have not been played then the order of contestant i and contestant $i+1$ cannot be determined, as the ranking order would still be consistent if we swapped the two.

Now given this information, we must finish the problem. There are 35 *critical* matches, and $\frac{36\cdot 35}{2} - 35 = \frac{35\cdot 34}{2}$ matches that aren't important to the final standings.

Each of the *critical* matches must be played. Now, we claim that each of the non-critical matches is played with probability exactly $\frac{35}{36}$. We can see this by looking at when each of these matches is played over all possible permutations of slips. The match is only not played in a given permutation if it is after all of the critical matches in that permutation, which happens with probability exactly $\frac{1}{36}$ by symmetry.

Now combining this information we get that the final answer is $35 + \frac{35\cdot 17\cdot 35}{36} = \frac{35\cdot(595+36)}{36} = \boxed{\frac{22085}{36}}$.

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9. For natural numbers n, let $r(n)$ be the number formed by reversing the digits of n, and take $f(n)$ to be the maximum value of $\frac{r(k)}{k}$ across all n-digit positive integers k.

If we define $g(n) = \left\lfloor \frac{1}{10-f(n)} \right\rfloor$, what is the value of $g(20)$?

Proposed by Adam Bertelli

Answer: 90909091

Solution. First, we will show that, for an even number of digits, the value of k maximizing $\frac{r(k)}{k}$ is $10\ldots 09\ldots 99$, where the two halves are equal in length. Note that the value of $\frac{r(k)}{k}$ for this number is less than 10. Now, if we decrease any digit on the right half, we are reducing the numerator of our expression at least 10 times faster than we are the denominator, so this will decrease the fraction. If we increase any digit on the left half, we are increasing the denominator faster than the numerator, so this will also decrease the fraction. Now, we can write our optimal number k as $10^{2n-1} + 10^n - 1$, where there are $2n$ digits, and $r(k)$ as $10^{2n} - 10^n + 1$. Thus,

$$\frac{1}{10-f(n)} = \frac{10^{2n-1}+10^n-1}{(10^{2n}+10^{n+1}-10)-(10^{2n}-10^n+1)} = \frac{10^{2n-1}+10^n-1}{10^{n+1}+10^n-11} = \frac{1}{11}\cdot\frac{10^{2n-1}+10^n-1}{10^n-1}$$

Note that $(10^n-1)(10^{n-1}+1) = (10^{2n-1}+10^n-1) - 10^{n-1}$, so our fraction is equal to

$$\frac{10^{n-1}+1}{11} + \frac{10^{n-1}}{11(10^n-1)}$$

As $n = 10$, the first term's numerator is in fact divisible by 11, and we can approximate the quotient by using the decimal expansion of $\frac{1}{11} = 0.090909\ldots$ to get that it must be the first integer greater than $90909090.909\ldots$, or $\boxed{90909091}$. The second term is approximately $\frac{1}{110}$, so we may safely discard it when flooring to get our answer.

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10. Adam places down cards one at a time from a standard 52 card deck (without replacement) in a pile. Each time he places a card, he gets points equal to the number of cards in a row immediately before his current card that are all the same suit as the current card. For instance, if there are currently two hearts on the top of the pile (and the third card in the pile is not hearts), then placing a heart would be worth 2 points, and placing a card of any other suit would be worth 0 points. What is the expected number of points Adam will have after placing all 52 cards?

Proposed by Adam Bertelli

Answer: $\dfrac{624}{41}$

Solution. We can observe that, due to the way the scoring system scales, our total score will in fact be equal to number of same-suit runs of length k in our deck over all $2 \le k \le 13$, including overlaps (for instance, the 4th card in a run will add an additional run of lengths 2, 3, and 4, and scores 3 additional points). Thus it suffices to compute the expected number of same-suit runs of length k. By linearity of expectation, we can simply compute the odds of a single run occurring within k fixed consecutive spots, and multiply this by the number of different places these k spots could be, which is $53 - k$.

The probability of a run occurring is 4 choices for fixing a suit, times $\binom{52-k}{39}$ ways to place the wrong suit cards outside of our run, divided by the total number of ways we could have placed our suit cards, or $\binom{52}{13}$. Thus our expression becomes $\frac{4}{\binom{52}{13}} \sum_{k=2}^{13} \binom{52-k}{39}(53-k)$. This is equivalent to $\frac{4}{\binom{52}{13}} \sum_{k=0}^{11} \binom{k+39}{39}(k+40)$, which is equivalent to $\frac{160}{\binom{52}{13}} \sum_{k=0}^{11} \binom{k+40}{40}$. By hockey-stick identity, this simplifies to $\frac{160}{\binom{52}{13}} \cdot \binom{52}{41} = \boxed{\dfrac{624}{41}}$.

11. Let $\{\varepsilon_i\}_{i\ge1}, \{\theta_i\}_{i\ge0}$ be two infinite sequences of real numbers, such that $\varepsilon_i \in \{-1, 1\}$ for all i, and the numbers θ_i obey

$$\tan\theta_{n+1} = \tan\theta_n + \varepsilon_n \sec(\theta_n) - \tan\theta_{n-1}, \qquad n \ge 1$$

and $\theta_0 = \frac{\pi}{4}, \theta_1 = \frac{2\pi}{3}$. Compute the sum of all possible values of

$$\lim_{m\to\infty} \left(\sum_{n=1}^{m} \frac{1}{\tan\theta_{n+1} + \tan\theta_{n-1}} + \tan\theta_m - \tan\theta_{m+1} \right)$$

Proposed by Grant Yu

Answer: $1 + \sqrt{3}$

Solution. Let $x_i = \tan\theta_i$. Then $x_0 = 1, x_1 = -\sqrt{3}$ and

$$x_{n+1} = x_n + \varepsilon_n \chi_n \sqrt{1 + x_n^2} - x_{n-1}$$

where $\chi_n := \frac{\sec(\theta_n)}{\sqrt{1+\tan^2\theta_n}} \in \{\pm1\}$.Thus

$$(x_{n+1} - x_n + x_{n-1})^2 = 1 + x_n^2.$$

Rearranging gives $(x_{n+1} + x_{n-1})^2 - 2(x_{n+1} + x_{n-1})x_n = 1$. Now, divide both sides by $x_{n-1} + x_{n+1}$ to get

$$\frac{1}{x_{n+1} + x_{n-1}} = x_{n-1} + x_{n+1} - 2x_n.$$

Therefore

$$\sum_{n=1}^{m} \frac{1}{\tan\theta_{n+1} + \tan\theta_{n-1}} + \tan\theta_m - \tan\theta_{m+1} = x_0 - x_1 = \boxed{1 + \sqrt{3}}$$

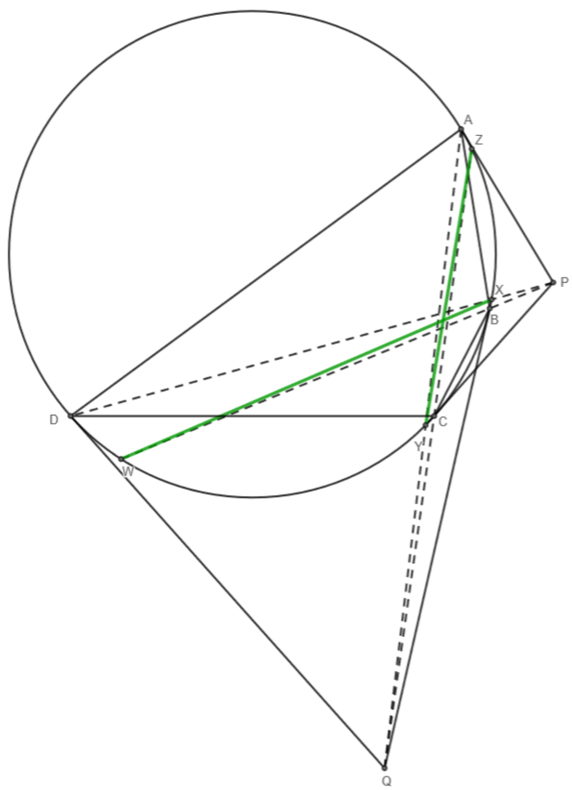
which is independent of m.

12. Let $ABCD$ be a cyclic quadrilateral with $AB = 3, BC = 2, CD = 6, DA = 8$, and circumcircle Γ. The tangents to Γ at A and C intersect at P and the tangents to Γ at B and D intersect at Q. Suppose lines PB and PD intersect Γ at points $W \neq B$ and $X \neq D$, respectively. Similarly, suppose lines QA and QC intersect Γ at points $Y \neq A$ and $Z \neq C$, respectively. What is the value of $\frac{WX^2}{YZ^2}$?

Proposed by Kyle Lee

Answer: $\dfrac{29}{14}$

Solution.



First, we compute $AC^2 = \frac{(ac+bd)(ad+bc)}{ab+cd} = \frac{68}{3}$ and $BD^2 = \frac{(ac+bd)(ab+cd)}{ad+bc} = 51$, where $a = AB = 3$, $b = BC = 2$, $c = CD = 6$, and $d = DA = 8$.

Lemma: Given a harmonic quadrilateral $PQRS$ with $PQ = p$, $QR = q$ and $PR = x$, we have

$$SP = p \cdot \frac{x}{\sqrt{2p^2 + 2q^2 - x^2}}$$

$$SR = q \cdot \frac{x}{\sqrt{2p^2 + 2q^2 - x^2}}.$$

To see this, first note that, by the definition of harmonic quadrilaterals, we have $\frac{QP}{QR} = \frac{SP}{SR}$, so we can write $SP = py$ and $SR = qy$ for some constant y. From here, applying the strong form of

Ptolemy's theorem yields

$$x^2 = \frac{2pqy \cdot (p^2+q^2)y}{pq(1+y^2)} \implies y^2 = \frac{x^2}{2(p^2+q^2)-x^2},$$

from which the desired conclusion directly follows.

Now, let $AC = m$ and $BD = n$ for ease of computation. Using the above lemma on $ABCW$, we have $WA = \frac{3m}{\sqrt{26-m^2}}$ and $WC = \frac{2m}{\sqrt{26-m^2}}$. Doing the same for $AXCD$, we get $XA = \frac{8m}{\sqrt{200-m^2}}$ and $XC = \frac{6m}{\sqrt{200-m^2}}$. Then by Ptolemy's theorem on $AXCW$, we have

$$\begin{aligned} WX^2 &= \left(\frac{AX \cdot CW + XC \cdot WA}{AC}\right)^2 \\ &= \left(\frac{\frac{34m^2}{\sqrt{(200-m^2)(26-m^2)}}}{m}\right)^2 \\ &= \frac{34^2m^2}{(200-m^2)(26-m^2)}. \end{aligned}$$

A similar approach yields $YZ^2 = \frac{34^2n^2}{(146-n^2)(80-n^2)}$. Finally, we compute

$$\begin{aligned} \frac{WX^2}{YZ^2} &= \frac{34^2m^2(146-n^2)(80-n^2)}{34^2n^2(200-m^2)(26-m^2)} \\ &= \frac{\frac{68}{3}(146-51)(80-51)}{51(200-\frac{68}{3})(26-\frac{68}{3})} \\ &= \boxed{\frac{29}{14}}. \end{aligned}$$

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13. Let F_n denote the nth Fibonacci number, with $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. There exists a unique two digit prime p such that for all n, $p | F_{n+100} + F_n$. Find p.

Proposed by Sam Rosenstrauch

Answer: 41

Solution. Let α and β be the roots of $x^2 - x - 1 = 0$. Then α^{50} and β^{50} are both roots of the polynomial $x^2 - (\alpha^{50} + \beta^{50})x + (\alpha\beta)^{50} = 0$, and from Vieta's we have $\alpha\beta = -1$ so we know α^{50} and β^{50} are roots of $x^2 - (\alpha^{50} + \beta^{50})x + 1 = 0$. However $\alpha^n + \beta^n$ is equal to the nth Lucas number, which are defined by $L_0 = 2, L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. Thus we have $\alpha^{100} - L_{50}\alpha^{50} + 1 = 0$ and $\beta^{100} - L_{50}\beta^{50} + 1 = 0$. Thus we have for all n $\alpha^{n+100} - L_{50}\alpha^{n+50} + \alpha^n = 0$(I) and $\beta^{n+100} - L_{50}\beta^{n+50} + \beta^n = 0$(II). Subtracting (I) from (II) and dividing both sides by $\alpha - \beta$ gives $\frac{\alpha^{n+100}-\beta^{n+100}}{\alpha-\beta} - L_{50} \cdot \frac{\alpha^{n+50}-\beta^{n+50}}{\alpha-\beta} + \frac{\alpha^n-\beta^n}{\alpha-\beta} = 0$. By Binet's formula $F_n = \frac{\alpha^n-\beta^n}{\alpha-\beta}$, so this gives $F_{n+100} + F_n = L_{50}F_{n+50}$. We now wish to examine the divisors of L_{50}. We see that it is the case that $\frac{L_{50}}{L_{10}} = \frac{\alpha^{50}+\beta^{50}}{\alpha^{10}+\beta^{10}} = \alpha^{40} - \alpha^{30}\beta^{10} + \alpha^{20}\beta^{20} - \alpha^{10}\beta^{30} + \beta^{40} = L_{40} - L_{20} + 1$, so $L_{10} | L_{50}$. We can compute $L_{10} = 123 = 3 \cdot 41$, which gives $\boxed{41}$ as our final answer.

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14. Let a tree on $mn+1$ vertices be (m,n)-nice if the following conditions hold:

- $m+1$ colors are assigned to the nodes of the tree
- for the first m colors, there will be exactly n nodes of color i $(1 \leq i \leq m)$
- the root node of the tree will be the unique node of color $m+1$.
- the (m,n)-nice trees must also satisfy the condition that for any two non-root nodes i, j, if the color of i equals the color of j, then the color of the parent of i equals the color of the parent of j.
- Nodes of the same color are considered indistinguishable (swapping any two of them results in the same tree).

Let $N(u,v,l)$ denote the number of (u,v)-nice trees with l leaves. Note that $N(2,2,2)=2, N(2,2,3)=4, N(2,2,4)=6$. Compute the remainder when $\sum_{l=123}^{789} N(8,101,l)$ is divided by 101.

Definition: Any rooted, ordered tree consists of some set of nodes, each of which has a (possibly empty) ordered list of children. Each node is the child of exactly one other node, with the exception of the root, which has no parent. There also cannot be any cycles of nodes which are all linearly children of each other.

Proposed by Advait Nene

Answer: 59

Solution. For convenience of discussion, draw the ordered tree in levels where within each level every vertex has the same distance to the root, and the vertices are drawn from left to right, with the root at the top and each level is a horizontal line below some level that is closer to the root. We proceed the discussion with $n = 101$ being a prime.

Consider the following operation on a (m,n)-nice tree: for each vertex in the tree we shift the edges going out of a vertex into edges going out of the vertex that has the same color and immediately to its right, if exists; otherwise shift these out-edges so that they become out-edges from the leftmost vertex that has the same color, keeping all endpoints of out-edges fixed throughout the operation. Note that all edges at the root node is fixed by the operation, while any other edge cycles through 101 vertices of a particular color, making its way back to its original position after we apply the operation 101 times. Let $T_1, \ldots, T_{101}$ be the results after we apply the operation $1, \ldots, 101$ times, respectively. Then T_{101} is the original tree and $T_1, \ldots, T_{101}$ are pairwise distinct, unless $T_1 = \cdots = T_{101}$ (since 101 is prime and the period of the orbit would divide 101).

It follows that the exceptions happen exactly when T_1 is a fixed point of the operation, i.e., every node of the same color must have the exact same children, and must all be exact copies of each other. Specifically leaves must appear in bundles of 101 as well, so if $N(m,n,k) \overset{101}{\not\equiv} 0$ then $101|k$. This also means that the edges between the root node and its closest vertices, along with which colors are connected to each other, determine a unique tree that has not been grouped into some group of 101 in an orbit produced by our operation. Let $N_s(m,n,k)$ be the number of (m,n)-nice trees with k leaves and s vertices in the first level among the exceptions. Then we may write $k = 101l$ for some nonnegative integer l and

$$N_s(m,101,k) = N_s(m,1,l := \frac{k}{101}) \left(\frac{\text{\# of ways to form 101s closest vertices to root}}{\text{\# of ways to form s closet vertices to root}} \right)$$

$$= \frac{\binom{101s}{101,\ldots,101}}{s!} N_s(m, 101, l).$$

Lemma: Thus $N(8, 101, 101l) \equiv N(8, 1, l) \pmod{101}$ and $N(8, 101, t) \equiv 0$ for $t \not\equiv 0 \pmod{101}$.

The case where t is not divisible by 101 has already been discussed above, so it suffices to prove the lemma when the number of leaves is divisible by 101.

Note that $\frac{1}{s!}\binom{101s}{101,\ldots,101}$ is also the number of partitions of $\mathbb{N}_{101s} = \{1, \ldots, 101s\}$ into s indistinguishable groups of 101 elements each. We do another shift by sending i to $i + 1 - 101 \cdot [101|i]$ where we use the notation $[S] = 1$ if the statement S is true and $[S] = 0$ if S is false for the indicator function, i.e. 1 gets sent to 2, 2 gets sent to 3 and so on, until 101 gets sent to $101 + 1 - 101 \cdot 1 = 1$ but 102 gets sent to $102 + 1 - 101 \cdot 0 = 103$. This shift is a bijection over $\mathbb{N}_{101s}$ with period dividing 101 (as before), so similar to the pairing argument before, in order to compute $\frac{1}{s!}\binom{101s}{101,\ldots,101}$ it suffices to compute the number of fixed points $\pmod{101}$. In this case we claim that there is only one fixed point when $s = 8$, namely $\{\{1, \ldots, 101\}, \ldots, \{101(s-1) + 1, \ldots, 101s\}\}$. Note that if a subset gets mapped to itself then all elements must be integers in some interval $(101t, 101(t + 1)]$. Otherwise this subset must get mapped to another subset in the partition. Repeatedly applying the shift, this subset must return to its original state after at most 8 moves (since there are at most $s = 8$ possibilities for elements of this subset before some state appears twice). However since each subset returns to its original state after 101 moves too and 101 is prime, the sequence of values the subset attains has period dividing both 101 and some number ≤ 8, which means that the subset repeats its value under our repeated shifts with period 1, contradiction. Thus all subsets are fixed by the shift as well, so we conclude that there is only one exception and

$$\frac{1}{s!}\binom{101s}{101, \ldots, 101} \equiv 1 \pmod{101}.$$

Coupled with the fact that $N_s(m, 101, k) = N_s(m, 101, l) \cdot \frac{1}{s!}\binom{101s}{101,\ldots,101}$ it follows that $N_s(m, 101, k) \equiv N_s(m, 101, l) \pmod{101}$, so recalling that we assumed $k = 101l$, we have

$$N(m, 101, 101l) = \sum_{s=0}^{m} N_s(m, 101, l) \equiv \sum_{s=0}^{m} N_s(m, 1, l) \equiv N(m, 1, l)$$

and the lemma follows. ■

Thus the problem is equivalent to $\sum_{l=2}^{7} N(8, 1, l) = 1372 \equiv \boxed{59} \pmod{101}$.

Solution 2: Let $G_{m,n}$ be the set of all (m, n)-nice trees. We can find the generating function for $N(m, n, k)$, for a fixed m. First, we find the generating function for a related set of trees, with the first layer of nodes all having the same color (i.e. the number of (m, n)-nice trees such that the root Call this set of trees $G_m^{(1)}$. Define the generating function $g_m(z, t)$, where the coefficient on the $z^n t^k$ term counts the number of trees in $G_m^{(1)}$ with k leaves.

Define the following functions: $W_m(k) = \frac{k!}{(m!)^{k/m}(k/m)!}$ and $\lambda(a, b, c) = \binom{a}{c}\binom{b-1}{a-c-1}$

We recursively create $G_m^{(1)}$ while keeping track of the number of leaves in each tree: take a set of r trees in $G_m^{(1)}$ of sizes having $n_1 + 1, n_2 + 1, \ldots, n_r + 1$ colors respectively with $n_1 + \cdots + n_r = n - 1$, remove their roots, then join the r groups of m nodes previously adjacent to some removed node with a new layer of m nodes, then connect the new layer to a new root in a way so that there are j leaves from the new layer for some $0 \leq j \leq m$ (which means that altogether the r trees we took from

$G_m^{(1)}$ had $k-j$ leaves if the new tree has k leaves) Since the colors are distinguishable it follows that the process is equivalent to choosing an ordered list of r trees then connect rm vertices to m vertices such that $m-j$ of the vertices have degree ≥ 1. The number of ways to join edges is $\binom{rm-1}{m-j-1}$ by stars and bars and there are $\binom{m}{j}$ ways to choose the vertices with nonzero degrees for a total of $\lambda(m, rm, j)$ ways to join vertices; there are $\frac{1}{r!}\binom{rm}{m} = W_m(rm)$ ways to partition a set of rm vertices into r groups of m to correspond to the layers adjacent to removed roots in the r trees taken from $G_m^{(1)}$, which means that the generating function for $G_m^{(1)}$-trees having k leaves given m nodes of each color satisfies $g_m(z,t) = z\ell_m(g_m(z,t),t)$ where $\ell_m(x,t) = t^m + \sum_{r=1}^{\infty} W_m(rm)x^r \sum_{j=0}^{m-1} \lambda(m, rm, j)t^j$. This encodes the recursive process, so we have Let the generating function for $N(m,n,k)$ be $G_m(z,t)$. We construct G_m by taking trees in $G_m^{(1)}$, breaking them apart at the root, shuffling around the resulting branches, and rejoining them at a new root. Using a similar argument as before (casing on r, the number of $G_m^{(1)}$-trees it takes to form the G_m-tree) we have

$$G_m(z,t) = \sum_{r=1}^{\infty} W_m(rm) g_m(z,t)^r$$

Now, we make the following claim:

$$N(p,n,k) \equiv \begin{cases} N(1, n, \frac{k}{p}) & p \mid k \\ 0 & p \nmid k \end{cases} \pmod p$$

We show that $G_p(z,t) \equiv G_1(z,t^p) \pmod p$. A quick calculation shows that $W_p(rp) \equiv 1 \pmod p$ for all r. Another calculation shows that $\lambda(p, rp, j) \equiv 0 \pmod p$ for $j > 0$, and for $j = 0$, $\lambda(p, rp, 0) \equiv 1 \pmod p$. $\implies G_p(z,t) \equiv G_1(z,t^p) \pmod p$ Now we can solve for $G_1(z,t)$.

$$G_1(z,t) = \frac{(1 - zt + z) + \sqrt{(z - zt - 1)^2 - 4zt}}{2z}$$

This is the generating function for the Narayana Numbers, which partition the Catalan Numbers.

$$\implies \sum_{k=303}^{606} N(101, 8, k) \equiv \sum_{k=2}^{7} N(1, 8, k) = 1372 \equiv \boxed{59} \pmod{101}.$$

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15. Let ABC be a triangle with $AB = 5, BC = 13$, and $AC = 12$. Let D be a point on minor arc AC of the circumcircle of ABC (endpoints excluded) and P on $\overline{BC}$. Let B_1, C_1 be the feet of perpendiculars from P onto CD, AB respectively and let BB_1, CC_1 hit (ABC) again at B_2, C_2 respectively. Suppose that D is chosen uniformly at random and AD, BC, B_2C_2 concur at a single point. Compute the expected value of BP/PC.

Proposed by Grant Yu

Answer: $\dfrac{1}{\arctan \frac{12}{5}}$

Solution. Set $\theta = \angle DCA$ and let $E = AB \cap CD$ then one can verify that $\frac{BP}{PC} = \frac{BE}{EC} = \frac{5\cos\theta + 12\sin\theta}{12}$. Integrating over θ gives the final answer of $\boxed{\dfrac{1}{\arctan \frac{12}{5}}}$.

Proof: We use the notation $(P_1P_2;P_3P_4) := \frac{P_1P_3}{P_1P_4} \cdot \frac{P_2P_4}{P_2P_3}$ (in signed lengths) to denote the *cross ratio* between the four points P_1, P_2, P_3, P_4 in that order.

Let the point of concurrence be K and note that $(AD;B_1C) \stackrel{B}{=} (ED;B_2C) \stackrel{K}{=} (DA;C_2B) \stackrel{C}{=} (EA;C_1B)$, hence C_1B_1 passes through K as well. Let H be the orthocenter of $\triangle EBC$ and F the foot of the altitude on BC. Then

$$-1 = (K(EF \cap AD); AD) \stackrel{E}{=} (K(EF \cap B_1C_1); B_1C_1) \stackrel{F}{=} (PE; B_1C_1)$$

so EP bisects $\angle BEC$, as desired.