CMIMC 2022
Team Round

INSTRUCTIONS

1. Do not look at the test before the proctor starts the round.
2. This test consists of 15 short-answer problems to be solved in 60 minutes.
3. Write your team name and team ID on your answer sheet.
4. Write your answers in the corresponding boxes on the answer sheets.
5. No computational aids other than pencil/pen are permitted.
6. If you believe that the test contains an error, submit your protest in writing to Doherty Hall 2302.

Team

1. Let $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$ be two squares. Construct 16 line segments A_iB_j for each possible $i, j \in \{1, 2, 3, 4\}$. What is the maximum number of line segments that don't intersect the edges of $A_1A_2A_3A_4$ or $B_1B_2B_3B_4$? (intersection with a vertex is not counted).

2. Find the smallest positive integer n for which $315^2 - n^2$ evenly divides $315^3 - n^3$.

3. Let $ABCD$ be a rectangle with $AB = 10$ and $AD = 5$. Suppose points P and Q are on segments CD and BC, respectively, such that the following conditions hold:

   - $BD \parallel PQ$
   - $\angle APQ = 90^\circ$.

   What is the area of $\triangle CPQ$?

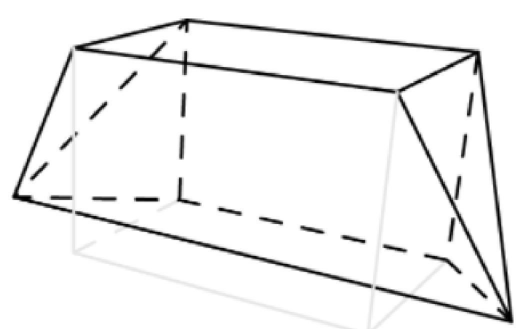
4. Let $\triangle ABC$ be equilateral with integer side length. Point X lies on $\overline{BC}$ strictly between B and C such that $BX < CX$. Let C' denote the reflection of C over the midpoint of $\overline{AX}$. If $BC' = 30$, find the sum of all possible side lengths of $\triangle ABC$.

5. For any integer a, let $f(a) = |a^4 - 36a^2 + 96a - 64|$. What is the sum of all values of $f(a)$ that are prime?

6. There are 9 points arranged in a 3×3 square grid. Let two points be adjacent if the distance between them is half the side length of the grid. (There should be 12 pairs of adjacent points)

   Suppose that we wanted to connect 8 pairs of adjacent points, such that all points are connected to each other. In how many ways is this possible?

7. A $3 \times 2 \times 2$ right rectangular prism has one of its edges with length 3 replaced with an edge of length 5 parallel to the original edge. The other 11 edges remain the same length, and the 6 vertices that are not endpoints of the replaced edge remain in place. The resulting convex solid has 8 faces, as shown below.

   Find the volume of the solid.

8. There are 36 contestants in the CMU Puyo-Puyo Tournament, each with distinct skill levels.

   The tournament works as follows: First, all $\binom{36}{2}$ pairings of players are written down on slips of paper and are placed in a hat. Next, a slip of paper is drawn from the hat, and those two players play a match. It is guaranteed that the player with a higher skill level will always win the match.

We continue drawing slips (without replacement) and playing matches until the results of the match completely determine the order of skill levels of all 36 contestants (i.e. there is only one possible ordering of skill levels consistent with the match results), at which point the tournament immediately finishes. What is the expected value of the number of matches played before the stopping point is reached?

9. For natural numbers n, let $r(n)$ be the number formed by reversing the digits of n, and take $f(n)$ to be the maximum value of $\frac{r(k)}{k}$ across all n-digit positive integers k.

   If we define $g(n) = \left\lfloor \frac{1}{10-f(n)} \right\rfloor$, what is the value of $g(20)$?

10. Adam places down cards one at a time from a standard 52 card deck (without replacement) in a pile. Each time he places a card, he gets points equal to the number of cards in a row immediately before his current card that are all the same suit as the current card. For instance, if there are currently two hearts on the top of the pile (and the third card in the pile is not hearts), then placing a heart would be worth 2 points, and placing a card of any other suit would be worth 0 points. What is the expected number of points Adam will have after placing all 52 cards?

11. Let $\{\varepsilon_i\}_{i\geq 1}, \{\theta_i\}_{i\geq 0}$ be two infinite sequences of real numbers, such that $\varepsilon_i \in \{-1, 1\}$ for all i, and the numbers θ_i obey

$$\tan\theta_{n+1} = \tan\theta_n + \varepsilon_n \sec(\theta_n) - \tan\theta_{n-1}, \qquad n \geq 1$$

and $\theta_0 = \frac{\pi}{4}, \theta_1 = \frac{2\pi}{3}$. Compute the sum of all possible values of

$$\lim_{m\to\infty}\left(\sum_{n=1}^{m} \frac{1}{\tan\theta_{n+1} + \tan\theta_{n-1}} + \tan\theta_m - \tan\theta_{m+1}\right)$$

12. Let $ABCD$ be a cyclic quadrilateral with $AB = 3, BC = 2, CD = 6, DA = 8$, and circumcircle Γ. The tangents to Γ at A and C intersect at P and the tangents to Γ at B and D intersect at Q. Suppose lines PB and PD intersect Γ at points $W \neq B$ and $X \neq D$, respectively. Similarly, suppose lines QA and QC intersect Γ at points $Y \neq A$ and $Z \neq C$, respectively. What is the value of $\frac{WX^2}{YZ^2}$?

13. Let F_n denote the nth Fibonacci number, with $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. There exists a unique two digit prime p such that for all n, $p | F_{n+100} + F_n$. Find p.

14. Let a tree on $mn + 1$ vertices be (m, n)-nice if the following conditions hold:

    - $m + 1$ colors are assigned to the nodes of the tree
    - for the first m colors, there will be exactly n nodes of color i ($1 \leq i \leq m$)
    - the root node of the tree will be the unique node of color $m + 1$.
    - the (m, n)-nice trees must also satisfy the condition that for any two non-root nodes i, j, if the color of i equals the color of j, then the color of the parent of i equals the color of the parent of j.
    - Nodes of the same color are considered indistinguishable (swapping any two of them results in the same tree).

Let $N(u, v, l)$ denote the number of (u, v)-nice trees with l leaves. Note that $N(2, 2, 2) = 2, N(2, 2, 3) = 4, N(2, 2, 4) = 6$. Compute the remainder when $\sum_{l=123}^{789} N(8, 101, l)$ is divided by 101.

Definition: Any rooted, ordered tree consists of some set of nodes, each of which has a (possibly empty) ordered list of children. Each node is the child of exactly one other node, with the exception of the root, which has no parent. There also cannot be any cycles of nodes which are all linearly children of each other.

15. Let ABC be a triangle with $AB = 5, BC = 13$, and $AC = 12$. Let D be a point on minor arc AC of the circumcircle of ABC (endpoints excluded) and P on $\overline{BC}$. Let B_1, C_1 be the feet of perpendiculars from P onto CD, AB respectively and let BB_1, CC_1 hit (ABC) again at B_2, C_2 respectively. Suppose that D is chosen uniformly at random and AD, BC, B_2C_2 concur at a single point. Compute the expected value of BP/PC.