Geometry Div. 2 Solutions

1. An equilateral 12-gon has side length 10 and interior angle measures that alternate between 90°, 90°, and 270°. Compute the area of this 12-gon.

   Proposed by Connor Gordon

   Answer: 500

   Solution: Note you can split the 12-gon into five congruent squares of side length 10, so the total area is $5(10^2) = \boxed{500}$.

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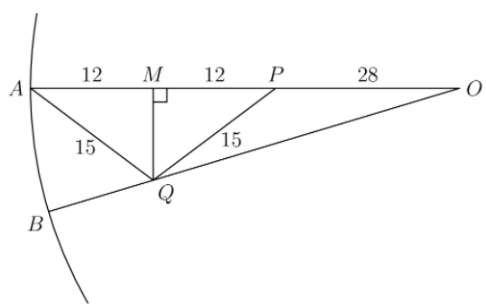
2. A circle has radius 52 and center O. Points A is on the circle, and point P on $\overline{OA}$ satisfies $OP = 28$. Point Q is constructed such that $QA = QP = 15$, and point B is constructed on the circle so that Q is on $\overline{OB}$. Find QB.

   Proposed by Justin Hsieh

   Answer: 11

   Solution:



   Let M be the midpoint of $\overline{AP}$. Then $AM = MP = \frac{52-28}{2} = 12$. Also, M is the altitude from Q of isosceles $\triangle AQP$, so $\angle QMP = \angle QMO = 90°$. We use the Pythagorean theorem on $\triangle QMP$ to get $QM = \sqrt{PQ^2 - MP^2} = \sqrt{15^2 - 12^2} = 9$. We use the Pythagorean theorem on $\triangle QMO$ to get $OQ = \sqrt{QM^2 + OM^2} = \sqrt{9^2 + 40^2} = 41$. Then $QB = OB - OQ = 52 - 41 = \boxed{11}$.

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3. Let ABC be an acute triangle with $\angle ABC = 60°$. Suppose points D and E are on lines AB and CB, respectively, such that CDB and AEB are equilateral triangles. Given that the positive difference between the perimeters of CDB and AEB is 60 and $DE = 45$, what is the value of $AB \cdot BC$?

   Proposed by Kyle Lee

   Answer: 1625

   Solution: Let r and s be the side lengths of CDB and AEB, respectively. Note that $AECD$ is an isosceles trapezoid, so by Ptolemy's theorem, we have

$$rs + (r-s)^2 = 45^2 \implies rs = 2025 - \left(\frac{60}{3}\right)^2 = 1625.$$

   Hence, $AB \cdot BC = \boxed{1625}$.

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4. Circle Γ has diameter $\overline{AB}$ with $AB = 6$. Point C is constructed on line AB so that $AB = BC$ and $A \neq C$. Let D be on Γ so that $\overleftrightarrow{CD}$ is tangent to Γ. Compute the distance from line $\overleftrightarrow{AD}$ to the circumcenter of $\triangle ADC$.

Proposed by Justin Hsieh

Answer: $4\sqrt{3}$

Solution: Let O be the center of Γ. If $\angle DOC = 2\theta$, then $\angle DOA = 180° - 2\theta$, so $\angle ODA = \angle OAD = \theta$. We can use the angle bisector theorem on $\angle O$ of $\triangle DOC$ to get that $\sin\theta = \frac{1}{\sqrt{3}}$.

Construct midpoint M of $\overline{AD}$, and let the perpendicular bisectors of $\overline{AD}$ and $\overline{AC}$ intersect at P. Note that P is the circumcenter of $\triangle ADC$. We have that $\triangle OMA \sim \triangle OBP$ by AA similarity, and we also have that $\frac{OM}{OA} = \sin\theta = \frac{1}{\sqrt{3}} = \frac{OB}{OP}$. Since the radius of Γ is 3, we get that $OM = \sqrt{3}$ and $OP = 3\sqrt{3}$.

Therefore, $MP = \boxed{4\sqrt{3}}$.

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5. Let ABC be an equilateral triangle of unit side length and suppose D is a point on segment $\overline{BC}$ such that $DB < DC$. Let M and N denote the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively. Suppose X and Y are the intersections of lines AB and ND, and lines AC and MD, respectively. Given that $XY = 4$, what is the value of $\frac{DB}{DC}$?

Proposed by Kyle Lee

Answer: $\dfrac{11-\sqrt{21}}{10}$

Solution: First, we show that Y lies on ray $\overrightarrow{DM}$ and X lies on ray $\overrightarrow{ND}$. Indeed, note that $BM = \frac{1}{2}$ and since there exists a point P on segment $\overline{BM}$ such that $BP < CP$, we have that line DM is steeper than line AC, so they intersect at Y past points M and A. Doing the same thing for X gives the desired.

Now observe that by an application of Menelaus' theorem, we have $AY = BX$. Then by the law of cosines, if x is this common length, we have

$$x^2 + (x+1)^2 - 2x(x+1)(\cos 120°) = 4^2 \implies x = \frac{1}{2}(\sqrt{21}-1).$$

Lastly, by Menelaus again, we have

$$\frac{x}{x+1} \cdot \frac{DC}{DB} \cdot 1 = 1 \implies \frac{DB}{DC} = \boxed{\frac{11-\sqrt{21}}{10}}.$$

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6. A triangle $\triangle ABC$ satisfies $AB = 13$, $BC = 14$, and $AC = 15$. Inside $\triangle ABC$ are three points X, Y, and Z such that:

- Y is the centroid of $\triangle ABX$
- Z is the centroid of $\triangle BCY$
- X is the centroid of $\triangle CAZ$

What is the area of $\triangle XYZ$?

Proposed by Adam Bertelli

Answer: $\dfrac{84}{13}$

Solution: Let's fix AC along $y=0$, and try to find the y coordinate of X, given the y coordinate of B is h. We have that $y(Y)=\frac{0+h+y(X)}{3}$, $y(Z)=\frac{0+h+y(Y)}{3}=\frac{4h+y(X)}{9}$, and $y(X)=\frac{0+0+y(Z)}{3}=\frac{4h+y(X)}{27}$, so $y(X)=\frac{2h}{13}$. Thus, if the area of $\triangle ABC$ is a, $[\triangle ACX]=\frac{2}{13}a$, and by symmetry $[\triangle ABY]=[\triangle BCZ]=\frac{2}{13}a$. Also, since connecting the centroid to the vertices divides a triangle into 3 triangles of equal area, we get that $[\triangle CZX]=[\triangle ACX]$, so we also have $[\triangle CZX]=[\triangle BYZ]=[\triangle AXY]=\frac{2}{13}a$. Now, subtracting off all 6 of these triangles from ABC leaves us with exactly $[\triangle XYZ]=\frac{a}{13}$, and since $a=84$ in a $13-14-15$ triangle, we get that the area is $\boxed{\dfrac{84}{13}}$.

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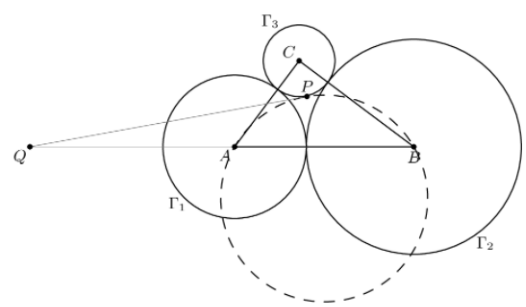
7. Let $\Gamma_1, \Gamma_2, \Gamma_3$ be three pairwise externally tangent circles with radii $1, 2, 3$, respectively. A circle passes through the centers of Γ_2 and Γ_3 and is externally tangent to Γ_1 at a point P. Suppose A and B are the centers of Γ_2 and Γ_3, respectively. What is the value of $\frac{PA^2}{PB^2}$?

Proposed by Kyle Lee

Answer: $\dfrac{8}{15}$

Solution:



Denote the new circle by ω, and suppose that its common tangent with Γ_1 intersects $\overline{AB}$ at Q. To begin, because $\overline{PQ}$ is the radical axis of ω and Γ_1, it must follow that Q has equal power with respect to both circles; equivalently,

$$\begin{aligned} QA\cdot QB = QC^2-1 &\implies QA\cdot(QA+5)=QC^2-1 \\ &\implies QA\cdot(QA+5)=\left((QA+\tfrac{9}{5})^2+(\tfrac{12}{5})^2\right)-1 \\ &\implies QA=\tfrac{40}{7}. \end{aligned}$$

To finish, we will use the following lemma.

Lemma: $\frac{PA^2}{PB^2}=\frac{QA}{QB}$
Proof: The simplest way to do this is by using similar triangles. Observe that $\triangle QAP\sim\triangle QPB$, so $\frac{QA}{QP}=\frac{QP}{QB}=\frac{PA}{PB}$. It follows that

$$\frac{QA}{QB}=\frac{QA}{QP}\cdot\frac{QP}{QB}=(\frac{PA}{PB})^2=\frac{PA^2}{PB^2},$$

as desired.

It follows that $\frac{PA^2}{PB^2}=\frac{QA}{QB}=\frac{40/7}{40/7+5}=\boxed{\frac{8}{15}}$.

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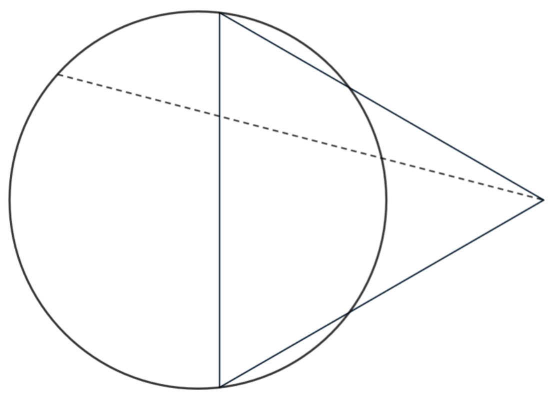
8. Let A and B be points on circle Γ such that $AB=\sqrt{10}$. Point C is outside Γ such that $\triangle ABC$ is equilateral. Let D be a point on Γ and suppose the line through C and D intersects AB and Γ

again at points E and $F \neq D$. It is given that points C, D, E, F are collinear in that order and that $CD = DE = EF$. What is the area of Γ?

Proposed by Kyle Lee

Answer: $\dfrac{38}{15}\pi$

Solution:



Let M be the midpoint of AB and suppose $CD = DE = EF = x$ and $EM = y$. Since ABC is equilateral, we know that $CM = \frac{\sqrt{3}}{2} \cdot \sqrt{10} = \frac{\sqrt{30}}{2}$. By the Pythagorean theorem, we must have

$$(2x)^2 = y^2 + \left(\frac{\sqrt{30}}{2}\right)^2 \implies 4x^2 = y^2 + \frac{15}{2}.$$

Moreover, by Power of a Point, we also have

$$x^2 = \left(\frac{\sqrt{10}}{2} - y\right)\left(\frac{\sqrt{10}}{2} + y\right) \implies 4x^2 = 10 - 4y^2.$$

Hence, $y^2 + \frac{15}{2} = 10 - 4y^2 \implies y^2 = \frac{1}{2}$, so $x^2 = \frac{10-4y^2}{4} = 2$. Now, since E is the midpoint of chord AB, we know that $\triangle CEO \sim \triangle CME$, where O is the center of Γ, so

$$CO = \frac{CE^2}{CM} = \frac{4x^2}{\frac{\sqrt{30}}{2}} = \frac{16}{\sqrt{30}}.$$

Lastly, the power of C wrt Γ is just $x(3x) = 3x^2$, so

$$3x^2 = CO^2 - r^2 \implies 3(2) = \frac{16^2}{30} - r^2 \implies r^2 = \frac{38}{15},$$

and the area of Γ is $\boxed{\dfrac{38}{15}\pi}$.