CMIMC 2022

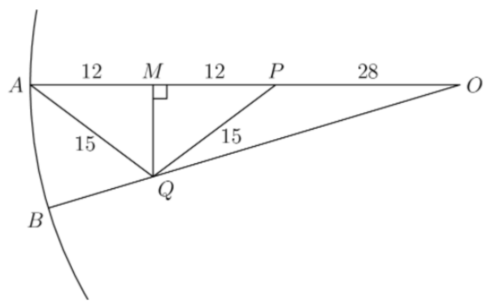
Geometry Div. 1 Solutions

1. A circle has radius 52 and center O. Points A is on the circle, and point P on $\overline{OA}$ satisfies $OP = 28$. Point Q is constructed such that $QA = QP = 15$, and point B is constructed on the circle so that Q is on $\overline{OB}$. Find QB.

   Proposed by Justin Hsieh

   Answer: 11

   Solution:



   Let M be the midpoint of $\overline{AP}$. Then $AM = MP = \frac{52-28}{2} = 12$. Also, M is the altitude from Q of isosceles $\triangle AQP$, so $\angle QMP = \angle QMO = 90°$. We use the Pythagorean theorem on $\triangle QMP$ to get $QM = \sqrt{PQ^2 - MP^2} = \sqrt{15^2 - 12^2} = 9$. We use the Pythagorean theorem on $\triangle QMO$ to get $OQ = \sqrt{QM^2 + OM^2} = \sqrt{9^2 + 40^2} = 41$. Then $QB = OB - OQ = 52 - 41 = \boxed{11}$.

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2. Let ABC be an acute triangle with $\angle ABC = 60°$. Suppose points D and E are on lines AB and CB, respectively, such that CDB and AEB are equilateral triangles. Given that the positive difference between the perimeters of CDB and AEB is 60 and $DE = 45$, what is the value of $AB \cdot BC$?

   Proposed by Kyle Lee

   Answer: 1625

   Solution: Let r and s be the side lengths of CDB and AEB, respectively. Note that $AECD$ is an isosceles trapezoid, so by Ptolemy's theorem, we have

$$rs + (r-s)^2 = 45^2 \implies rs = 2025 - \left(\frac{60}{3}\right)^2 = 1625.$$

   Hence, $AB \cdot BC = \boxed{1625}$.

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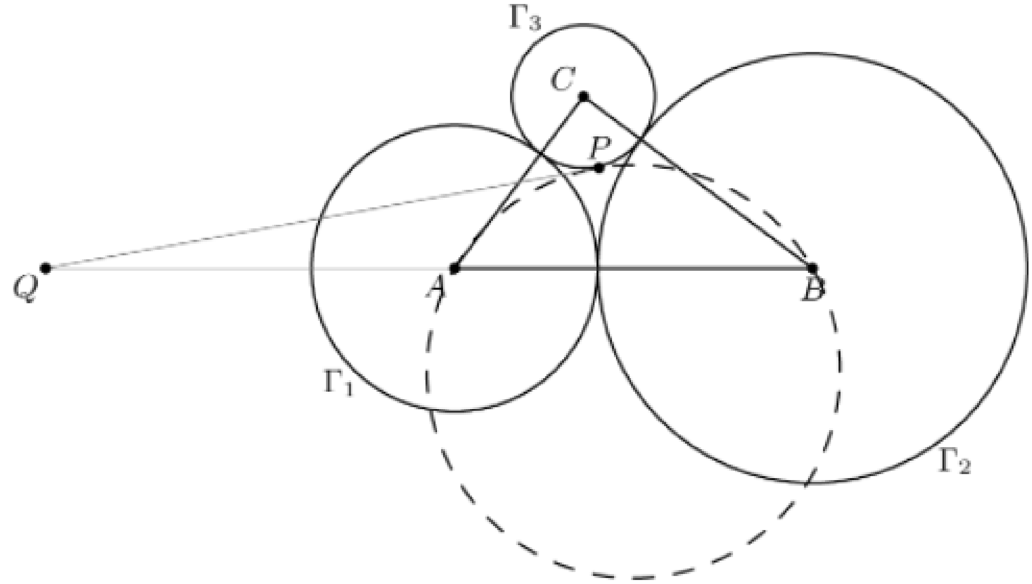
3. Let $\Gamma_1, \Gamma_2, \Gamma_3$ be three pairwise externally tangent circles with radii $1, 2, 3$, respectively. A circle passes through the centers of Γ_2 and Γ_3 and is externally tangent to Γ_1 at a point P. Suppose A and B are the centers of Γ_2 and Γ_3, respectively. What is the value of $\frac{PA^2}{PB^2}$?

   Proposed by Kyle Lee

   Answer: $\dfrac{8}{15}$

   Solution:



Denote the new circle by ω, and suppose that its common tangent with Γ_1 intersects $\overline{AB}$ at Q. To begin, because $\overline{PQ}$ is the radical axis of ω and Γ_1, it must follow that Q has equal power with respect to both circles; equivalently,

$$\begin{aligned} QA \cdot QB = QC^2 - 1 &\implies QA \cdot (QA + 5) = QC^2 - 1 \\ &\implies QA \cdot (QA + 5) = \left((QA + \tfrac{9}{5})^2 + (\tfrac{12}{5})^2\right) - 1 \\ &\implies QA = \tfrac{40}{7}. \end{aligned}$$

To finish, we will use the following lemma.

Lemma: $\frac{PA^2}{PB^2} = \frac{QA}{QB}$
Proof: The simplest way to do this is by using similar triangles. Observe that $\triangle QAP \sim \triangle QPB$, so $\frac{QA}{QP} = \frac{QP}{QB} = \frac{PA}{PB}$. It follows that

$$\frac{QA}{QB} = \frac{QA}{QP} \cdot \frac{QP}{QB} = (\frac{PA}{PB})^2 = \frac{PA^2}{PB^2},$$

as desired.

It follows that $\frac{PA^2}{PB^2} = \frac{QA}{QB} = \frac{40/7}{40/7+5} = \boxed{\frac{8}{15}}$.

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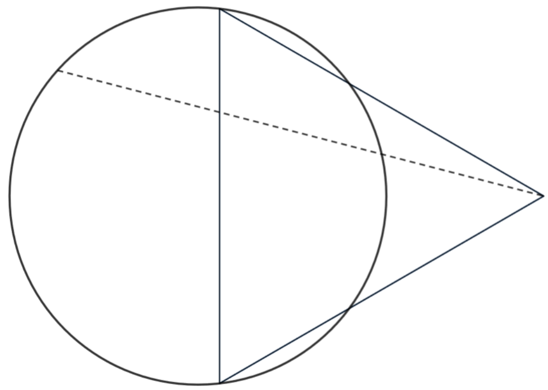
4. Let A and B be points on circle Γ such that $AB = \sqrt{10}$. Point C is outside Γ such that $\triangle ABC$ is equilateral. Let D be a point on Γ and suppose the line through C and D intersects AB and Γ again at points E and $F \neq D$. It is given that points C, D, E, F are collinear in that order and that $CD = DE = EF$. What is the area of Γ?

   Proposed by Kyle Lee

   Answer: $\dfrac{38}{15}\pi$

   Solution:



Let M be the midpoint of AB and suppose $CD = DE = EF = x$ and $EM = y$. Since ABC is equilateral, we know that $CM = \frac{\sqrt{3}}{2} \cdot \sqrt{10} = \frac{\sqrt{30}}{2}$. By the Pythagorean theorem, we must have

$$(2x)^2 = y^2 + \left(\frac{\sqrt{30}}{2}\right)^2 \implies 4x^2 = y^2 + \frac{15}{2}.$$

Moreover, by Power of a Point, we also have

$$x^2 = \left(\frac{\sqrt{10}}{2} - y\right)\left(\frac{\sqrt{10}}{2} + y\right) \implies 4x^2 = 10 - 4y^2.$$

Hence, $y^2 + \frac{15}{2} = 10 - 4y^2 \implies y^2 = \frac{1}{2}$, so $x^2 = \frac{10-4y^2}{4} = 2$. Now, since E is the midpoint of chord AB, we know that $\triangle CEO \sim \triangle CME$, where O is the center of Γ, so

$$CO = \frac{CE^2}{CM} = \frac{4x^2}{\frac{\sqrt{30}}{2}} = \frac{16}{\sqrt{30}}.$$

Lastly, the power of C wrt Γ is just $x(3x) = 3x^2$, so

$$3x^2 = CO^2 - r^2 \implies 3(2) = \frac{16^2}{30} - r^2 \implies r^2 = \frac{38}{15},$$

and the area of Γ is $\boxed{\frac{38}{15}\pi}$.

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5. In triangle ABC, let I, O, H be the incenter, circumcenter and orthocenter, respectively. Suppose that $AI = 11$ and $AO = AH = 13$. Find OH.

Proposed by Kevin You

Answer: 10

Solution: Let M be the midpoint of BC, and N be the midpoint of arc BC. Of course, $ON = AO = AH = R$, and $AH \parallel ON$, so $AONH$ is a parallelogram.

Lemma. $2OM = AH$. See https://www.cut-the-knot.org/triangle/ReflectionsOfOrthocenter.shtml.

Now, BC and ON bisects each other, which means $OBNC$ is a parallelogram. So, $\triangle OBN, \triangle OCN$ are equilateral. It follows that $\angle BOC = 120°$ and $\angle A = 60°$.

However, $\angle BIC = 90 + \angle A/2 = 120°$ as well, so $BIOC$ is cyclic. More specifically, I, O, B, C all have distance R from N.

Since A, I, N are colinear, we write the parallelogram law,

$$AO^2 + ON^2 + NH^2 + HA^2 = AN^2 + HO^2$$

$$4R^2 = (AI + R)^2 + HO^2$$

This solves to $HO = \boxed{10}$.

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6. Let Γ_1 and Γ_2 be two circles with radii r_1 and r_2, respectively, where $r_1 > r_2$. Suppose Γ_1 and Γ_2 intersect at two distinct points A and B. A point C is selected on ray $\overrightarrow{AB}$, past B, and the tangents to Γ_1 and Γ_2 from C are marked as points P and Q, respectively. Suppose that Γ_2 passes through the center of Γ_1 and that points P, B, Q are collinear in that order, with $PB = 3$ and $QB = 2$. What is the length of AB?

Proposed by Kyle Lee

Answer: $\dfrac{3\sqrt{14}}{2}$

Solution: (Author)
Let C_1 and C_2 denote the centers of Γ_1 and Γ_2, respectively. By a homothety, we can easily see that there exists a constant c such that $C_1B = 3c$ and $C_2B = 2c$. Moreover, since Γ_2 passes through the center of Γ_1, we also have $C_1C_2 = 2c$. By the Law of Cosines, we have

$$(2c)^2 = (3c)^2 + (2c)^2 - 2(3c)(2c)\cos\angle C_1BC_2 \Rightarrow \cos\angle C_1BC_2 = \frac{-3}{4}.$$

Now, remark that since P, B, Q are collinear (with $PQ = 3 + 2 = 5$), $CP = CQ$, and moreover $PC_1B \sim BC_2Q$, we have

$$\angle BPC = \angle BQC = 90^\circ - \angle C_1PB = 90^\circ - \frac{180^\circ - \angle C_1BC_2}{2} = \frac{\angle C_1BC_2}{2}.$$

In particular, $\cos\angle BPC = \sqrt{\dfrac{1 + 3/4}{2}} = \sqrt{\dfrac{7}{8}}$, which implies

$$PC^2 = \frac{(3+2)^2/4}{7/8} = \frac{50}{7}.$$

Now, we have

$$BC^2 = \frac{50}{7} - (5/2)^2 + (1/2)^2 = \frac{8}{7},$$

so by Power of a Point on Γ_1, we have

$$PC^2 = CB \cdot CA = CB(CB + BA)$$

$$\implies \frac{50}{7} = \sqrt{\frac{8}{7}}\left(\sqrt{\frac{8}{7}} + AB\right)$$

$$\implies AB = \sqrt{\frac{7}{8}}\left(\frac{50}{7} - \frac{8}{7}\right)$$

$$\implies AB = \boxed{\frac{3\sqrt{14}}{2}}.$$

Solution 2: (Kevin You)
Claim 1. $APCQ$ is cyclic. Furthermore, C is midpoint of arc PQ.

$\angle CBQ = \angle CQA$, $\angle CBP = \angle CPA$ by tangency. But $\angle CBP$ and $\angle CBQ$ are supplementary, therefore so are $\angle CPA$ and $\angle CBQ$. So, $APCQ$ is cyclic. Furthermore, by radical axis $CP^2 = CB \cdot CA = CQ^2$, so C is midpoint of arc PQ.

Claim 2. I_2, the incenter of ABQ lies on Γ_1.

Since A, B are on O_1, A, B are equidistant to O_1. However, O_1 lies on Γ_2, which means that O_1 is on the midpoint of arc AB (of Γ_2). By the incenter-excenter lemma, I_2 therefore is also equidistant to O_1, hence I_2 is on Γ_1.

Claim 3. $AQ = PQ$.

Letting $\angle AQP = \theta$, we have $\angle AI_2B = 90^\circ + \theta/2$. Now, AI_2BP is cyclic, so $APQ = 90^\circ - \theta/2$. So, $\triangle AQP$ is isosceles.

We have $AQ = 5$, by angle bisector theorem $AP = \frac{15}{2}$. We finish with Stewart's theorem on $\triangle APQ$.

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7. In acute $\triangle ABC$, let I denote the incenter and suppose that line AI intersects segment BC at a point D. Given that $AI = 3, ID = 2$, and $BI^2 + CI^2 = 64$, compute BC^2.

Proposed by Kyle Lee

Answer: $\dfrac{272}{3}$

Solution: Without loss of generality, suppose $AB < AC$. Let F be the foot of I onto BC and let $E = AI \cap (BAC)$. Note that $\angle DCE = \angle BAE = \angle CAE$, so $\triangle DCE \sim \triangle DAC$. If we let $CD = l$, we have by Fact 5 that $DE = l - 2$, so $\frac{l}{l-2} = \frac{l+3}{l}$, whence $l = CD = 6$ and $DE = 4$. Now, let M be the midpoint of BC and suppose $BM = CM = x$. Note that $EM = \sqrt{6^2 - x^2}$ and $DM = \sqrt{4^2 - EM^2} = \sqrt{4^2 - (6^2 - x^2)} = \sqrt{x^2 - 20}$. Moreover, $\triangle IDF \sim \triangle EDM$, so $DF = \frac{\sqrt{x^2-20}}{2}$ and $IF = \frac{\sqrt{6^2-x^2}}{2}$. We have that

$$\begin{aligned} BI^2 + CI^2 &= BF^2 + FC^2 + 2IF^2 \\ &= (x - \frac{3}{2}\sqrt{x^2-20})^2 + (x + \frac{3}{2}\sqrt{x^2-20})^2 + \frac{6^2 - x^2}{2} \\ &= 2\left(x^2 + \frac{9}{4}(x^2 - 20)\right) + \frac{6^2 - x^2}{2} \\ &= 6x^2 - 72. \end{aligned}$$

Hence,

$$6x^2 - 72 = 64 \implies BC^2 = (2x)^2 = \frac{2}{3}(72 + 64) = \boxed{\frac{272}{3}}.$$

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8. Let $ABCD$ be a cyclic quadrilateral with circumcenter O. Rays $\overrightarrow{OB}$ and $\overrightarrow{DC}$ intersect at E, and rays $\overrightarrow{OC}$ and $\overrightarrow{AB}$ intersect at F. Suppose that $AE = EC = CF = 4$, and the circumcircle of ODE bisects $\overline{BF}$. Find the area of triangle ADF.

Proposed by Howard Halim

Answer: $\dfrac{9\sqrt{7}}{2}$

Solution: Since $EA = EC$ and $OA = OC$, points A and C must be symmetric wrt line EO. Therefore, EO is the angle bisector of $\angle AEC$. Since O also lies on the perpendicular bisector of AD, it is the midpoint of arc AD on the circumcircle of AED. By Fact 5, B must be the incenter of AED, since it lies on segment EO (the E-angle bisector) and the circle centered at O passing through A and D.

Let M be the midpoint of BF, which lies on $(AODE)$. Since B is the incenter, AB is the A-angle bisector of AED, and M is the midpoint of arc DE on $(AODE)$. Therefore, by Fact 5 again, F must be the A-excenter of ADE.

This means that $EO \perp EF$, because they are the E-internal and E-external angle bisectors of AED. Since $CE = CF$, C is on the perpendicular bisector of EF. But C is also on the hypotenuse of right triangle OEF, so C must be the midpoint of OF. This means that $OC = CF = 4$, so $OA = OB = OD = 4$ as well, since they are the radius of the same circle.

Let F' be the reflection of F across OE. Then F also lies on lines BC and AO (by symmetry w.r.t. OE). Since C and E are the midpoints of OF and FF', OE and $F'C$ are medians of triangle OFF', and their intersection point B is the centroid of $\triangle OFF'$. Therefore, $BE = \frac{1}{2}BO = 2$ and $OE = 6$.

Since $AE = 4 = OD$ and $AODE$ is cyclic, $AODE$ must be an isosceles trapezoid, so it's diagonals have equal length: $AD = OE = 6$. If we let P be the intersection of AB with DE, then by the angle bisector theorem, $DP : PE = DA : AE = 3 : 2$, so

$$[ADF] = \frac{3}{2}[AEF]$$

Since B is the centroid of $\triangle OFF'$, $AB : BF = 1 : 2$, so

$$[AEF] = \frac{3}{2}[BEF]$$

But $EF = \sqrt{OF^2 - OE^2} = 2\sqrt{7}$, so $[BEF] = 2\sqrt{7}$ and

$$[ADF] = \frac{3}{2} \cdot \frac{3}{2} \cdot 2\sqrt{7} = \boxed{\frac{9\sqrt{7}}{2}}$$