CMIMC 2022

Combinatorics and Computer Science Round

INSTRUCTIONS

1. Do not look at the test before the proctor starts the round.
2. This test consists of 10 short-answer problems to be solved in 60 minutes and one estimation question. Each of the short-answer questions is worth points depending on its difficulty, and the estimation question will be used to break ties. **If you do not write an estimate for estimation, you will be placed last in tiebreaking.**
3. Write your name, team name, and team ID on your answer sheet. Circle the subject of the test you are taking.
4. Write your answers in the corresponding boxes on the answer sheets.
5. No computational aids other than pencil/pen are permitted.
6. Answers must be reasonably simplified.
7. If you believe that the test contains an error, submit your protest in writing to Doherty 2302 by the middle of events (5:15 PM).

1. A particle starts at $(0,0,0)$ in three-dimensional space. Each second, it randomly selects one of the eight lattice points a distance of $\sqrt{3}$ from its current location and moves to that point. What is the probability that, after two seconds, the particle is a distance of $2\sqrt{2}$ from its original location?

   Proposed by Connor Gordon

   Answer: $\frac{3}{8}$

   Solution. Note that $8 = 4+4+0$ has a unique representation as a sum of 3 squares (up to order) which means that the fly has one of the coordinates being 0 and the other two at ± 2. Note that the change in each coordinate is independent of the others and the absolute value of each coordinate becomes 0 or 2 with probability $\frac{1}{2}$ each after 2 seconds. There are 3 ways this could happen, namely $(\pm 2, \pm 2, 0), (\pm 2, 0, \pm 2), (0, \pm 2, \pm 2)$, giving the answer $\boxed{\dfrac{3}{8}}$.

---

2. Starting with a 5×5 grid, choose a 4×4 square in it. Then, choose a 3×3 square in the 4×4 square, and a 2×2 square in the 3×3 square, and a 1×1 square in the 2×2 square. Assuming all squares chosen are made of unit squares inside the grid. In how many ways can the squares be chosen so that the final 1×1 square is the center of the original 5×5 grid?

   Proposed by Nancy Kuang

   Answer: 36

   Solution. We can view each successive selection as removing a row (either bottom-most or top-most) and a column (either left-most or right-most) from our grid. Out of the 4 steps in each dimension, we want to take away the top row twice, and the leftmost row twice. Thus the total number of orders we can do this is $\binom{4}{2}^2 = \boxed{36}$, since we can choose which row/column to remove independent of one another.

---

3. We say that a set S of 3 unit squares is *commutable* if $S = \{s_1, s_2, s_3\}$ for some s_1, s_2, s_3 where s_2 shares a side with each of s_1, s_3. How many ways are there to partition a 3×3 grid of unit squares into 3 pairwise disjoint commutable sets?

   Proposed by Srinivasan Sathiamurthy

   Answer: 10

   Solution. We say that a set is a *row set* if its three squares form a row in the 3×3 grid and define *column set* similarly. A key observation is that there are no configuration with both a row set and a column set, but each valid partition must have at least one row set or column set. Another observation we could make is that if such a set is in the middle row/column then the other 2 rows/columns are forced to be row/column sets. Thus the possible configurations are all rows, all columns, or we fix one of the top row, bottom row, left column, or right column, and form the rest out of L shapes in 2 possible ways, giving $2 + 4 \cdot 2 = \boxed{10}$ total partitions.

---

4. Dilhan is running around a track for 12 laps. If halfway through a lap, Dilhan has his phone on him, he has a $\frac{1}{3}$ chance to drop it there. If Dilhan runs past his phone on the ground, he will attempt to

pick it up with a $\frac{2}{3}$ chance of success, and won't drop it for the rest of the lap. He starts with his phone at the start of the 5K, what is the chance he still has it when he finished the 5K?

Proposed by Zack Lee, Daniel Li, Dilhan Salgado

Answer: $\frac{2}{3}$

Solution. Regardless of whether or not he started the lap with a phone, after the process of running past the halfway point, Dilhan has a $\boxed{\frac{2}{3}}$ chance of having the phone.

---

5. Daniel, Ethan, and Zack are playing a multi-round game of Tetris. Whoever wins 11 rounds first is crowned the champion. However Zack is trying to pull off a "reverse-sweep", where (at-least) one of the other two players first hits 10 wins while Zack is still at 0, but Zack still ends up being the first to reach 11. How many possible sequences of round wins can lead to Zack pulling off a reverse sweep?

Proposed by Dilhan Salgado

Answer: $2\binom{31}{10} - \binom{20}{10}$

Solution. First let's count the number of ways for Zack to Reverse Sweep Daniel. We know Daniel wins 10 games and then Zack wins 11 games. We then need to count the number of ways to add in the games Ethan wins. This is equivalent to the number of ways to order 21(DZ)s and 10(E)s. The first 10 'DZ' letters correspond to the Daniel wins and last 11 correspond to the Zack wins. The positions of the 'E's correspond to the Ethan wins ('E's after the final Zack correspond to the cases where Ethan wins less than 10 rounds when the game ends). Thus there are $\binom{31}{10}$ ways for Zack to sweep Daniel.

Similarly, there are $\binom{31}{10}$ ways for Zack to reverse sweep Ethan. However, we need to account for the fact that we have counted cases where Zack sweeps both other players twice. These cases are when both other players reach 10 wins, and then Zack wins 11 straight rounds. There are $\binom{20}{10}$ ways to order these Daniel/Ethan wins.

Thus subtracting out the overcount, we get the total possible reverse sweep sequences is $\boxed{2\binom{31}{10} - \binom{20}{10}}$.

---

6. A sequence of pairwise distinct positive integers is called averaging if each term after the first two is the average of the previous two terms. Let M be the maximum possible number of terms in an averaging sequence in which every term is less than or equal to 2022 and let N be the number of such distinct sequences (every term less than or equal to 2022) with exactly M terms. What is $M + N$? (Two sequences $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ are said to be distinct if $a_i \neq b_i$ for some integer $1 \leq i \leq n$).

Proposed by Kyle Lee

Answer: 2008

Solution. Observe that if the first two terms differ in absolute value by d, then the second and third terms differ in absolute value by $d/2$, the third and fourth terms differ in absolute value by $d/4$, and so on.

Then to make the integer sequence as large as possible, the first two terms should differ by the largest power of two possible. Since $2^{10} < 2022 - 1 < 2^{11}$, the difference between the first two terms should be 2^{10}. The sequence stops when two consecutive terms differ by 1, which occurs between the 11th and 12th terms. Thus, $M = 10 + 2 = 12$.

Now, note that any such sequence is determined by the first two terms. Since we want the first two terms to differ by 1024, we require the first two terms to be equivalent modulo 1024. It is easy to see that $\{1, 1025\}, \{2, 1026\}, \cdots, \{997, 2021\}, \{998, 2022\}$ are all possible unordered pairs, so $N = 2 \cdot 998 = 1996$. The desired answer is $12 + 1996 = \boxed{2008}$.

---

7. For a family gathering, 8 people order one dish each. The family sits around a circular table. Find the number of ways to place the dishes so that each person's dish is either to the left, right, or directly in front of them.

   Proposed by Nicole Sim

   Answer: 49

   Solution. Number the people $1, \ldots, 8$. If two people sitting next to each other have both their dishes placed in the same (not on front of them) direction, then everyone must have their dishes placed in that direction, forming a cycle. This contributes 2 placements.

   For the remaining cases, note that each person either has his dish in front of him or the person has his dish swapped with one of his neighbors. Thus the question becomes counting the number of ways to partition a circle of 8 points into some singletons and adjacent pairs.

   If person 1 is not in a pair with anyone else then this becomes equivalent to finding the number of ways to do this with 7 points on a line, which is easy to work out as $F_7 (= 21)$. Otherwise 1 is paired with 8 or 2 and in each case it becomes pairing 6 points on a line, for a total of $2 \cdot F_6$ ways. This gives the answer $F_7 + 2F_6 + 2 = \boxed{49}$.

---

8. The CMU Kiltie Band is attempting to crash a helicopter via grappling hook. The helicopter starts parallel (angle 0 degrees) to the ground. Each time the band members pull the hook, they tilt the helicopter forward by either x or $x + 1$ degrees, with equal probability, if the helicopter is currently at an angle x degrees with the ground. Causing the helicopter to tilt to 90 degrees or beyond will crash the helicopter. Find the expected number of times the band must pull the hook in order to crash the helicopter.

   Proposed by Justin Hsieh

   Answer: $\frac{269}{32}$

   Solution. After 1 turn, the helicopter has angle 0 or 1 with the ground, each with equal probability. After 2 turns, the helicopter has angle 0, 1, 2, or 3 with the ground, each with equal probability. This is because an angle of 0 can change to 0 or 1, and an angle of 1 can change to 2 or 3. In general, for $0 \leq k \leq 2^n - 1$, the helicopter has an angle k with the ground after n turns with probability $\frac{1}{2^n}$. In particular, after 6 turns, the helicopter could have any integer angle from 0 to 63 degrees, each with equal probability.

   If the helicopter's angle is at least 45 degrees, then the next pull will guarantee that the helicopter crashes. If the helicopter's angle is no more than 44 degrees, the helicopter will have an angle no more than 89 degrees after the next pull. The seventh pull will result in a crash with probability $\frac{19}{64}$, since there are 19 integers from 45 to 63, inclusive. With probability $1-\frac{19}{64}=\frac{45}{64}$, the seventh pull will result in a tilt of 0 to 89 degrees, distributed equally in probability. (This is because the tilt is in 0 through 44 before the pull.) In this state, there is a $\frac{1}{2}$ probability that the next pull will crash the helicopter, and a $\frac{1}{2}$ probability that the tilt will be 0 to 89 again.

Once the angle of the helicopter is 0 to 89, there is a $\frac{1}{2}$ that the next pull will crash the helicopter, so the expected number of pulls it takes to crash the helicopter is $\frac{1}{1/2}=2$. Starting from 0 degrees, the expected number of pulls to crash the helicopter is $\frac{19}{64}(7)+\frac{45}{64}(7+2)=\left(\frac{19}{64}+\frac{45}{64}\right)\cdot 7+\frac{45}{64}\cdot 2=7+\frac{45}{32}=\boxed{\dfrac{269}{32}}$.