CMIMC 2022

Combinatorics and Computer Science Round

INSTRUCTIONS

1. Do not look at the test before the proctor starts the round.
2. This test consists of 10 short-answer problems to be solved in 60 minutes and one estimation question. Each of the short-answer questions is worth points depending on its difficulty, and the estimation question will be used to break ties. **If you do not write an estimate for estimation, you will be placed last in tiebreaking.**
3. Write your name, team name, and team ID on your answer sheet. Circle the subject of the test you are taking.
4. Write your answers in the corresponding boxes on the answer sheets.
5. No computational aids other than pencil/pen are permitted.
6. Answers must be reasonably simplified.
7. If you believe that the test contains an error, submit your protest in writing to Doherty 2302 by the middle of events (5:15 PM).

Combinatorics and Computer Science

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1. A particle starts at $(0,0,0)$ in three-dimensional space. Each second, it randomly selects one of the eight lattice points a distance of $\sqrt{3}$ from its current location and moves to that point. What is the probability that, after two seconds, the particle is a distance of $2\sqrt{2}$ from its original location?

2. Starting with a 5×5 grid, choose a 4×4 square in it. Then, choose a 3×3 square in the 4×4 square, and a 2×2 square in the 3×3 square, and a 1×1 square in the 2×2 square. Assuming all squares chosen are made of unit squares inside the grid. In how many ways can the squares be chosen so that the final 1×1 square is the center of the original 5×5 grid?

3. We say that a set S of 3 unit squares is *commutable* if $S = \{s_1, s_2, s_3\}$ for some s_1, s_2, s_3 where s_2 shares a side with each of s_1, s_3. How many ways are there to partition a 3×3 grid of unit squares into 3 pairwise disjoint commutable sets?

4. Dilhan is running around a track for 12 laps. If halfway through a lap, Dilhan has his phone on him, he has a $\frac{1}{3}$ chance to drop it there. If Dilhan runs past his phone on the ground, he will attempt to pick it up with a $\frac{2}{3}$ chance of success, and won't drop it for the rest of the lap. He starts with his phone at the start of the 5K, what is the chance he still has it when he finished the 5K?

5. Daniel, Ethan, and Zack are playing a multi-round game of Tetris. Whoever wins 11 rounds first is crowned the champion. However Zack is trying to pull off a "reverse-sweep", where (at-least) one of the other two players first hits 10 wins while Zack is still at 0, but Zack still ends up being the first to reach 11. How many possible sequences of round wins can lead to Zack pulling off a reverse sweep?

6. A sequence of pairwise distinct positive integers is called averaging if each term after the first two is the average of the previous two terms. Let M be the maximum possible number of terms in an averaging sequence in which every term is less than or equal to 2022 and let N be the number of such distinct sequences (every term less than or equal to 2022) with exactly M terms. What is $M + N$? (Two sequences $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ are said to be distinct if $a_i \neq b_i$ for some integer $1 \leq i \leq n$).

7. For a family gathering, 8 people order one dish each. The family sits around a circular table. Find the number of ways to place the dishes so that each person's dish is either to the left, right, or directly in front of them.

8. The CMU Kiltie Band is attempting to crash a helicopter via grappling hook. The helicopter starts parallel (angle 0 degrees) to the ground. Each time the band members pull the hook, they tilt the helicopter forward by either x or $x+1$ degrees, with equal probability, if the helicopter is currently at an angle x degrees with the ground. Causing the helicopter to tilt to 90 degrees or beyond will crash the helicopter. Find the expected number of times the band must pull the hook in order to crash the helicopter.