CMIMC 2022

Combinatorics and Computer Science Round

INSTRUCTIONS

1. Do not look at the test before the proctor starts the round.
2. This test consists of 10 short-answer problems to be solved in 60 minutes and one estimation question. Each of the short-answer questions is worth points depending on its difficulty, and the estimation question will be used to break ties. **If you do not write an estimate for estimation, you will be placed last in tiebreaking.**
3. Write your name, team name, and team ID on your answer sheet. Circle the subject of the test you are taking.
4. Write your answers in the corresponding boxes on the answer sheets.
5. No computational aids other than pencil/pen are permitted.
6. Answers must be reasonably simplified.
7. If you believe that the test contains an error, submit your protest in writing to Doherty 2302 by the middle of events (5:15 PM).

1. Starting with a 5×5 grid, choose a 4×4 square in its interior. Then, choose a 3×3 square in the 4×4 square, and a 2×2 square in the 3×3 square, and a 1×1 square in the 2×2 square. Assuming all squares chosen are made of unit squares inside the grid. In how many ways can the squares be chosen so that the final 1×1 square is the center of the original 5×5 grid?

   Proposed by Nancy Kuang

   Answer: 36

   Solution. We can view each successive selection as removing a row (either bottom-most or top-most) and a column (either left-most or right-most) from our grid. Out of the 4 steps in each dimension, we want to take away the top row twice, and the leftmost row twice. Thus the total number of orders we can do this is $\binom{4}{2}^2 = \boxed{36}$, since we can choose which row/column to remove independent of one another.

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2. A sequence of pairwise distinct positive integers is called averaging if each term after the first two is the average of the previous two terms. Let M be the maximum possible number of terms in an averaging sequence in which every term is less than or equal to 2022 and let N be the number of such distinct sequences (every term less than or equal to 2022) with exactly M terms. What is $M + N$? (Two sequences $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ are said to be distinct if $a_i \neq b_i$ for some integer $1 \leq i \leq n$).

   Proposed by Kyle Lee

   Answer: 2008

   Solution. Observe that if the first two terms differ in absolute value by d, then the second and third terms differ in absolute value by $d/2$, the third and fourth terms differ in absolute value by $d/4$, and so on.

   Then to make the integer sequence as large as possible, the first two terms should differ by the largest power of two possible. Since $2^{10} < 2022 - 1 < 2^{11}$, the difference between the first two terms should be 2^{10}. The sequence stops when two consecutive terms differ by 1, which occurs between the 11th and 12th terms. Thus, $M = 10 + 2 = 12$.

   Now, note that any such sequence is determined by the first two terms. Since we want the first two terms to differ by 1024, we require the first two terms to be equivalent modulo 1024. It is easy to see that $\{1, 1025\}, \{2, 1026\}, \cdots, \{997, 2021\}, \{998, 2022\}$ are all possible unordered pairs, so $N = 2 \cdot 998 = 1996$. The desired answer is $12 + 1996 = \boxed{2008}$.

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3. For a family gathering, 8 people order one dish each. The family sits around a circular table. Find the number of ways to place the dishes so that each person's dish is either to the left, right, or directly in front of them.

   Proposed by Nicole Sim

   Answer: 49

   Solution. Number the people $1, \ldots, 8$. If two people sitting next to each other have both their dishes placed in the same (not on front of them) direction, then everyone must have their dishes placed in that direction, forming a cycle. This contributes 2 placements.

For the remaining cases, note that each person either has his dish in front of him or the person has his dish swapped with one of his neighbors. Thus the question becomes counting the number of ways to partition a circle of 8 points into some singletons and adjacent pairs.

If person 1 is not in a pair with anyone else then this becomes equivalent to finding the number of ways to do this with 7 points on a line, which is easy to work out as $F_7(= 21)$. Otherwise 1 is paired with 8 or 2 and in each case it becomes pairing 6 points on a line, for a total of $2 \cdot F_6$ ways. This gives the answer $F_7 + 2F_6 + 2 = \boxed{49}$.

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4. The CMU Kiltie Band are attempting to crash a helicopter via grappling hook. The helicopter starts parallel (angle 0 degrees) to the ground. Each time the band members pull the hook, they tilt the helicopter forward by either x or $x + 1$ degrees, with equal probability, if the helicopter is currently at an angle x degrees with the ground. Causing the helicopter to tilt to 90 degrees or beyond will crash the helicopter. Find the expected number of times the band must pull the hook in order to crash the helicopter.

   Proposed by Justin Hsieh

   Answer: $\frac{269}{32}$

   Solution. After 1 turn, the helicopter has angle 0 or 1 with the ground, each with equal probability. After 2 turns, the helicopter has angle 0, 1, 2, or 3 with the ground, each with equal probability. This is because an angle of 0 can change to 0 or 1, and an angle of 1 can change to 2 or 3. In general, for $0 \le k \le 2^n - 1$, the helicopter has an angle k with the ground after n turns with probability $\frac{1}{2^n}$. In particular, after 6 turns, the helicopter could have any integer angle from 0 to 63 degrees, each with equal probability.

   If the helicopter's angle is at least 45 degrees, then the next pull will guarantee that the helicopter crashes. If the helicopter's angle is no more than 44 degrees, the helicopter will have an angle no more than 89 degrees after the next pull. The seventh pull will result in a crash with probability $\frac{19}{64}$, since there are 19 integers from 45 to 63, inclusive. With probability $1 - \frac{19}{64} = \frac{45}{64}$, the seventh pull will result in a tilt of 0 to 89 degrees, distributed equally in probability. (This is because the tilt is in 0 through 44 before the pull.) In this state, there is a $\frac{1}{2}$ probability that the next pull will crash the helicopter, and a $\frac{1}{2}$ probability that the tilt will be 0 to 89 again.

   Once the angle of the helicopter is 0 to 89, there is a $\frac{1}{2}$ that the next pull will crash the helicopter, so the expected number of pulls it takes to crash the helicopter is $\frac{1}{1/2} = 2$. Starting from 0 degrees, the expected number of pulls to crash the helicopter is $\frac{19}{64}(7) + \frac{45}{64}(7 + 2) = \left(\frac{19}{64} + \frac{45}{64}\right) \cdot 7 + \frac{45}{64} \cdot 2 = 7 + \frac{45}{32} = \boxed{\frac{269}{32}}$.

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5. At CMIMC headquarters, there is a row of n lightbulbs, each of which is connected to a light switch. Daniel the electrician knows that exactly one of the switches doesn't work, and needs to find out which one. Every second, he can do exactly one of 3 things:

   - Flip a switch, changing the lightbulb from off/on or on/off (unless the switch is broken).
   - Check if a given lightbulb is on or off.
   - Measure the total electricity usage of all the lightbulbs, which tells him exactly how many are currently on.

Initially, all the lightbulbs are off. Daniel was given the very difficult task of finding the broken switch in at most n seconds, but fortunately he showed up to work 10 seconds early today. What is the largest possible value n such that he can complete his task on time?

Proposed by Adam Bertelli

Answer: 4084

Solution. Note that operation 1 is an action, while operations 2 and 3 are queries. If we perform the same query twice without any action in between, it will give the same result, so it makes sense to think of our process as a sequence of actions, followed by a sequence of queries, followed by a sequence of actions, and so on. The first sequence of actions must be of the form "turn on $n-k$ lightbulbs" for some k.

Let $f(n)$ be the smallest number of steps it takes to find the broken lightbulb among n lightbulbs. Then, after our initial action, the only meaningful query we can make is operation 3, to see if the broken lightbulb is in our set of $n-k$ or not (performing operation 2 is equivalent to doing this on a set of size 1). If it is, then it is most efficient to just check $n-k-1$ of the lightbulbs to see which one is off (if none of them are, the last one is). This is optimal, because if we perform less than $n-k-1$ queries, there exist two lightbulbs in our set of $n-k$ that cannot be distinguished from one another, so if one of them is broken, we cannot determine it. On the other hand, if it is not in our set, the number of remaining steps we need to find it is simply $f(k)$, hence we get that

$$f(n) = \min_{0<k<n}[n-k+1+\max(n-k-1, f(k))]$$

Since we are interested in how much $f(n)$ exceeds n, let's define $g(n) = f(n) - n$, giving us the analogous relation

$$g(n) = \min_{0<k<n}[1+\max(n-2k-1, g(k))]$$

Now, let g_m be the largest value such that $g(g_m) = m$. Then, we know that $g(g_{m+1}) = 1 + g(g_m)$, thus when we set $k = g_m$, the expression $1 + \max(g_{m+1} - 2k - 1, g(k))$ must equal $1 + g(g_m)$, as any larger value of k would produce a strictly larger output. This tells us that, when g_{m+1} is as large as possible, it satisfies $g_{m+1} = 2g_m + m + 1$. Starting from $g_{(-1)} = 1$, it is not hard to derive that $g_m = 2^{m+2} - m - 2$, so the value we are looking for is $g_{10} = 2^{12} - 12 = \boxed{4084}$.

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6. Barry has a standard die containing the numbers 1-6 on its faces.

   He rolls the die continuously, keeping track of the sum of the numbers he has rolled so far, starting from 0. Let E_n be the expected number of time he needs to until his recorded sum is at least n.

   It turns out that there exist positive reals a, b such that

$$\lim_{n\to\infty} E_n - (an+b) = 0$$

   Find (a, b).

   Proposed by Dilhan Salgado

   Answer: $(\frac{2}{7}, \frac{10}{21})$

Solution. Let V_n be the expected value of Barry's sum the first time it reaches n. By Linearity of Expectation, we can see that $V_n = \frac{7}{2}E_n$.

Now we just want to compute for large enough n the distribution of the first value that Barry reaches that is at least n (Which we denote as X_n).. It is clear that this number will be in the set $\{n, n+1, n+2, n+3, n+4, n+5\}$. We can easily see that in the limit Barry reaches each number with probability $\frac{2}{7}$. Now for Barry to reach $n+5$ as the first number $\geq n$ he must get to exactly $n-1$ (probability $\frac{2}{7}$) and then roll exactly a 6 (probability $\frac{1}{6}$), so in total, the probability is $\frac{2}{42} = \frac{1}{21}$. Similarly for $n+4$ he must roll a 6 from $n-2$ or a 5 from $n-1$. These two event will each occur in $\frac{1}{21}$ of all possible roll sequences. Thus $P(X_n = n+4) = \frac{2}{21}$).

As the pattern clearly continues, we can see that $P(X_n = n+i) = \frac{6-i}{21}$. Thus $V_n = E[X_n] = \frac{6n+5(n+1)+4(n+2)+3(n+3)+2(n+4)+1(n+5)}{21} = n + \frac{5+8+9+8+5}{21} = n + \frac{5}{3}$ (note that this works only in the limit case, as we need the probability of hitting each number to be $\frac{2}{7}$).

We can now calculate that in the limit $E_n = \frac{2}{7}V_n = \frac{2}{7}n + \frac{10}{21}$, so our answer is $\boxed{\left(\frac{2}{7}, \frac{10}{21}\right)}$

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7. In a class of 12 students, no two people are the same height. Compute the total number of ways for the students to arrange themselves in a line such that:

   - for all $1 < i < 12$, the person in the i-th position (with the leftmost position being 1) is taller than exactly $i \pmod 3$ of their adjacent neighbors, and
   - the students standing at positions which are multiples of 3 are strictly increasing in height from left to right.

Proposed by Nancy Kuang

Answer: 63700

Solution 1. (Grant Yu) We count using a process that generates all such sequences: let $12 = 3n$, start with number i written at position $3i$. On step $k(\geq 1)$, write two distinct positive integers taken from $\{n-k+1, ..., n+2k\}$ at slots $3(n-k)+1, 3(n-k)+2$ in increasing order and increase an original copy of a number by 1 when a new duplicate appears. Note that the numbers newly written cannot be the pair $(n-k+1, n-k+2)$ (or else slot $3(n-k)+3$ has a number that's bigger than what's written on slot $3(n-k)+2$), and any other choice of the numbers written evidently work, for a total of $3k(3k-1)/2-1 = (3k+1)(3k-2)/2$. Multiplying over all k gives the answer of $\boxed{63700}$.

Example: when $3n = 12$, $n = 4$ so we start with __1__2__3__4 and need to choose a pair of numbers from $\{4, 5, 6\}$ other than $(4, 5)$ at slots $3(n-k)+1, 3(n-k)+2 = 10, 11$, e.g. $(4, 6)$, then the numbers written become __1__2__346$(4+1)$ = __1__2__3465. Next we choose a pair of numbers from $\{3, 4, 5, 6, 7, 8\}$ other than $(3, 4)$, e.g. $(4, 5)$, then the numbers written become __1__2453$(4+1)$6$(5+1)$ $\to$ __1__2453$(5+1)(6+1)$6 $\to$ __1__245367$(6+1)$ $\to$ __1__24536$(7+1)$7 $\to$ __1__2453687. In the next step we choose a pair from $\{2, 3, \ldots, 10\}$ other than $(2, 3)$ to insert in the rightmost slots that are still empty and in the last step we choose from $\{1, 2, \ldots, 12\}$ other than the pair $(1, 2)$ to fill in the remaining two slots.

Solution 2. (Nancy Kuang) Work backwards, Let p_i be the height of person i. Start with $p_3 < p_6 < p_9 < p_{12}$. Consider the pair p_{10}, p_{11}. We must have $p_9 < p_{10} < p_{11}$ and $p_{12} < p_{11}$. We have three cases:

Case 1: $p_{10} < p_{12}$. This gives 1 solution.

Case 2: p_{10}, p_{11} adjacent. This gives 1 solution.

Case 3: $p_{12} < p_{10}$, and p_{10}, p_{11} not adjacent. This gives $\binom{1}{2} = 0$ solutions.

Next, place p_7, p_8, then p_4, p_5, then p_1, p_2. Note that we must have $p_{3(4-i)} < p_{3(4-i)-2} < p_{3(4-i)-1}$ and $p_{3(4-i)} < p_{3(4-i)-1}$, and $3i$ of the people that have been placed so far are taller than person $3(4-i)$. Using the same cases as above, we have

Case 1: $p_{3(4-i)-2} < p_{3(4-i)}$. There are $3i+1$ ways to place person $p_{3(4-i)-1}$.

Case 2: $p_{3(4-i)-2}, p_{3(4-i)-1}$ adjacent. There are $3i+1$ ways to place the pair.

Case 3: $p_{3(4-i)} < p_{3(4-i)-2}$, and $p_{3(4-i)-2}, p_{3(4-i)-1}$ not adjacent. There are $\binom{3i+1}{2} = \frac{3i(3i+1)}{2}$ ways to choose the two spots such that there is someone between the pair.

Adding the three, we get $\frac{(3i+1)(3i+4)}{2}$.

Since placing each pair is independent, we can multiply to get $\frac{(1\cdot 4)(4\cdot 7)(7\cdot 10)(10\cdot 13)}{2^4} = \boxed{63700}$

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8. Daniel has a (mostly) standard deck of 54 cards, consisting of 4 suits each containing the ranks 1 to 13 as well as 2 jokers.

   Daniel plays the following game: He shuffles the deck uniformly randomly and then takes all of the cards that end up strictly between the two jokers. He then sums up the ranks of all the cards he has taken and calls that his score.

   Let p be the probability that his score is a multiple of 13. There exists relatively prime positive integers a and b, with b as small as possible, such that $|p - a/b| < 10^{-10}$. What is a/b?

   Proposed by Dilhan Salgado, Daniel Li

   Answer: $\frac{77}{689}$

   Solution. Letting $f(x, y) = \prod_{i=1}^{13}(1 + x^i y)^4$ and define $g(y) := \frac{1}{13}\sum_{k=1}^{13} f(\exp 2\pi i k/13, y)$, we compute

$$g(y) = \frac{1}{13}((1+y)^{52} + (1+y^{13})^4 \sum_{k=1}^{12}(1 + y\exp(2\pi i k/13))^4) = \frac{1}{13}((1+y)^{52} + (1+y^{13})^4 \cdot 12)$$

   so the coefficient g_m of x^m in $g(y)$ is the number of ways to choose m cards from the 54 card deck, none of which are jokers, with sum is divisible by 13. There are 2 ways to place the jokers on their sides and $53 - m$ ways to place the block consisting of 2 jokers and m cards inside, along with $m!(52-m)!$ ways to order these cards, so

$$p = \frac{1}{54!}\sum_{m=0}^{52} 2\cdot(53-m)!m!\cdot g_m$$

$$= \frac{2}{54!}\sum_{m=0}^{52}(53-m)!m!(\binom{52}{m} + [13|m]\binom{4}{\lfloor\frac{m}{13}\rfloor})$$

$$= \frac{2}{54!}\sum_{m=0}^{52} 52(53-m) + \sum_{k=0}^{4}\frac{2}{54!}(53-13k)!(13k)!\binom{4}{k}.$$

We note that $\frac{1}{54!}(53-13k)! = \frac{1}{54\cdot 52!}\cdot\frac{53-13k}{53}\cdot(52-13k)! < \frac{1}{54\cdot 52!}(52-13k)!$ so that the 2nd sum

$$\sum_{k=0}^{4}\frac{2}{54!}(53-13k)!(13k)!\binom{4}{k} \le \frac{2}{54!}\left(53!0!\binom{4}{0}+1!52!\binom{4}{4}\right)+\frac{32}{54\binom{52}{13}}(*)$$

but the 2nd sum is also greater than its value summed over just $k=0,4$ which are first 2 terms in the right hand side of the inequality $(*)$. We also bound

$$\binom{52}{13} \ge \binom{52}{13}^2\cdot\frac{50}{11}\cdot\frac{49\cdot 48\cdot 47\cdot 46\cdot 45\cdot 44\cdot 43\cdot 42\cdot 41\cdot 40}{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}$$

Note that $\frac{49\cdot 48\cdot 47\cdot 46}{10\cdot 9\cdot 8\cdot 7} = 7\cdot 6\cdot\frac{47\cdot 46}{10\cdot 9} > 42\cdot 25 = 1050 > 2^{10}$ and $\frac{45\cdot 44\cdot 43\cdot 42}{6\cdot 5\cdot 4\cdot 3} = \frac{3}{2}\cdot 11\cdot 43\cdot 14 = 11\cdot 43\cdot 21 > 8192 = 2^{13}$, $\frac{41\cdot 40}{2\cdot 1} > \frac{2^{10}}{5}$, hence the RHS is greater than $4^2\cdot\frac{50}{11}\cdot 2^{10}\cdot 2^{13}\cdot 2^{10}/5 = \frac{2^{37}\cdot 10}{11}$, so the third term

$$\frac{32}{54\binom{52}{13}} \le \frac{32}{54}\cdot\frac{2^{-37}\cdot 11}{10} < 2^{-37} < 10^{-11}.$$

Thus,

$$\boxed{\frac{77}{689}} = \frac{2}{54!}\left(\sum_{m=0}^{52}(53-m)!m!\binom{52}{m}+\sum_{k=0,4}(53-13k)!(13k)!\binom{4}{k}\right) < p < \frac{77}{689}+10^{-11}.$$

Now suppose $|p-\frac{a}{b}| < 10^{-10}$ and b is as small as possible. Then by triangle inequality $|\frac{a}{b}-\frac{77}{689}| \le 10^{-10}+10^{-11} < 10^{-9}$ which means that $10^9 \le |689a-77b|10^9 < 689b$, contradiction to $b < 689$.