CMIMC 2022

Combinatorics and Computer Science Round

INSTRUCTIONS

1. Do not look at the test before the proctor starts the round.
2. This test consists of 10 short-answer problems to be solved in 60 minutes and one estimation question. Each of the short-answer questions is worth points depending on its difficulty, and the estimation question will be used to break ties. **If you do not write an estimate for estimation, you will be placed last in tiebreaking.**
3. Write your name, team name, and team ID on your answer sheet. Circle the subject of the test you are taking.
4. Write your answers in the corresponding boxes on the answer sheets.
5. No computational aids other than pencil/pen are permitted.
6. Answers must be reasonably simplified.
7. If you believe that the test contains an error, submit your protest in writing to Doherty 2302 by the middle of events (5:15 PM).

1. Starting with a 5×5 grid, choose a 4×4 square in it. Then, choose a 3×3 square in the 4×4 square, and a 2×2 square in the 3×3 square, and a 1×1 square in the 2×2 square. Assuming all squares chosen are made of unit squares inside the grid. In how many ways can the squares be chosen so that the final 1×1 square is the center of the original 5×5 grid?

2. A sequence of pairwise distinct positive integers is called averaging if each term after the first two is the average of the previous two terms. Let M be the maximum possible number of terms in an averaging sequence in which every term is less than or equal to 2022 and let N be the number of such distinct sequences (every term less than or equal to 2022) with exactly M terms. What is $M + N$? (Two sequences $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ are said to be distinct if $a_i \neq b_i$ for some integer $1 \leq i \leq n$).

3. For a family gathering, 8 people order one dish each. The family sits around a circular table. Find the number of ways to place the dishes so that each person's dish is either to the left, right, or directly in front of them.

4. The CMU Kiltie Band are attempting to crash a helicopter via grappling hook. The helicopter starts parallel (angle 0 degrees) to the ground. Each time the band members pull the hook, they tilt the helicopter forward by either x or $x + 1$ degrees, with equal probability, if the helicopter is currently at an angle x degrees with the ground. Causing the helicopter to tilt to 90 degrees or beyond will crash the helicopter. Find the expected number of times the band must pull the hook in order to crash the helicopter.

5. At CMIMC headquarters, there is a row of n lightbulbs, each of which is connected to a light switch. Daniel the electrician knows that exactly one of the switches doesn't work, and needs to find out which one. Every second, he can do exactly one of 3 things:

   - Flip a switch, changing the lightbulb from off/on or on/off (unless the switch is broken).
   - Check if a given lightbulb is on or off.
   - Measure the total electricity usage of all the lightbulbs, which tells him exactly how many are currently on.

   Initially, all the lightbulbs are off. Daniel was given the very difficult task of finding the broken switch in at most n seconds, but fortunately he showed up to work 10 seconds early today. What is the largest possible value n such that he can complete his task on time?

6. Barry has a standard die containing the numbers 1-6 on its faces.

   He rolls the die continuously, keeping track of the sum of the numbers he has rolled so far, starting from 0. Let E_n be the expected number of time he needs to until his recorded sum is at least n.

   It turns out that there exist positive reals a, b such that

   $$\lim_{n\to\infty} E_n - (an + b) = 0$$

   Find (a, b).

7. In a class of 12 students, no two people are the same height. Compute the total number of ways for the students to arrange themselves in a line such that:

   - for all $1 < i < 12$, the person in the i-th position (with the leftmost position being 1) is taller than exactly $i \pmod 3$ of their adjacent neighbors, and

- the students standing at positions which are multiples of 3 are strictly increasing in height from left to right.

8. Daniel has a (mostly) standard deck of 54 cards, consisting of 4 suits each containing the ranks 1 to 13 as well as 2 jokers.

   Daniel plays the following game: He shuffles the deck uniformly randomly and then takes all of the cards that end up strictly between the two jokers. He then sums up the ranks of all the cards he has taken and calls that his score.

   Let p be the probability that his score is a multiple of 13. There exists relatively prime positive integers a and b, with b as small as possible, such that $|p - a/b| < 10^{-10}$. What is a/b?