CMIMC 2022

Algebra and Number Theory Round

INSTRUCTIONS

1. Do not look at the test before the proctor starts the round.
2. This test consists of 10 short-answer problems to be solved in 60 minutes and one estimation question. Each of the short-answer questions is worth points depending on its difficulty, and the estimation question will be used to break ties. **If you do not write an estimate for estimation, you will be placed last in tiebreaking.**
3. Write your name, team name, and team ID on your answer sheet. Circle the subject of the test you are taking.
4. Write your answers in the corresponding boxes on the answer sheets.
5. No computational aids other than pencil/pen are permitted.
6. Answers must be reasonably simplified.
7. If you believe that the test contains an error, submit your protest in writing to Doherty 2302 by the end of lunch.

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Algebra and Number Theory

1. Alice and Bob live on the same road. At time t, they both decide to walk to each other's houses at constant speed. However, they were busy thinking about math so that they didn't realize passing each other. Alice arrived at Bob's house at $3:19$pm, and Bob arrived at Alice's house at $3:29$pm. Charlie, who was driving by, noted that Alice and Bob passed each other at $3:11$pm. Find the difference in minutes between the time Alice and Bob left their own houses and noon on that day.

   Proposed by Kevin You

   Answer: 179

   Solution. We work in differences of minutes with respect to the passing. Suppose that the events occurred at time $-t, 0, 8, 18$ minutes. Without loss of generality, let the distance between the houses be d.

   Then, Alice's speed is $\frac{d}{t+8}$, and Bob's speed is $\frac{d}{t+18}$. The relative speed between them is $\frac{d}{t+18} + \frac{d}{t+8}$. Since the relative distance between Alice and Bob is d at time $-t$, and 0 at time 0, we must have that the relative speed between Alice and Bob is $\frac{d}{t}$.

   Combining the two expressions, we obtain

$$\frac{d}{t+18} + \frac{d}{t+8} = \frac{d}{t}$$

   This solves to

$$\begin{aligned} t \cdot (t+18) + t \cdot (t+8) &= (t+18) \cdot (t+8) \\ 2t^2 + (18+8)t &= t^2 + (18+8)t + 18 \cdot 8 \\ t^2 &= 18 \cdot 8 \\ t &= 12 \end{aligned}$$

   So, t occurs 12 minutes before $3:11$pm, or $t = 2:59$pm, which is $\boxed{179}$ minutes after noon.

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2. Arthur, Bob, and Carla each choose a three-digit number. They each multiply the digits of their own numbers. Arthur gets 64, Bob gets 35, and Carla gets 81. Then, they add corresponding digits of their numbers together. The total of the hundreds place is 24, that of the tens place is 12, and that of the ones place is 6. What is the difference between the largest and smallest of the three original numbers?

   Proposed by Jacob Weiner

   Answer: 182

   Solution. Arthur's digits must all be powers of two. He has 6 "credits" to spend where 1 costs 0, 2 costs 1, 4 costs 2, and 8 costs 3. Bob's digits must be a permutation of 157. Carla's digits must be powers of three, similar to Arthur's, having 4 credits.

   The sum of three one-digit numbers is 24 if and only if at least one number is greater than or equal to 8. It is either Arthur's (8) or Carla's (9). If Arthur's hundreds is 8, then the remaining numbers must be 7,9. If Carla's hundreds is 9, then the remaining numbers must be 7,8. So the hundreds digits are A8, B7, C9. Hence Arthur has 3 credits remaining, Bob has 5 and 1, and Carla has 2 credits.

Bob's 5 cannot be in the ones place because that would force the other digits to be 1,0 in some order – and no product is zero. So Bob is 751.

The remaining 7 in the tens place must be a sum of a power of two and power of three. The only way this can happen is 4+3. Hence the numbers are A842, B751, C933

The largest is 933, and the smallest is 751, so the difference is $\boxed{182}$.

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3. How many 4-digit numbers have exactly 9 divisors from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$?

Proposed by Ethan Gu

Answer: 33

Solution. We can quickly see that 1, 3, 4 and 5 will be required if we want to reach 9 divisors. This also gives us 2, 6, and 10. This means we can choose two from the remaining three numbers: 7, 8, and 9 to reach exactly 9 divisors. This gives us:

$8 \times 9 \times 5 = 360$ (omits 7)

$8 \times 3 \times 7 \times 5 = 840$ (omits 9)

$4 \times 9 \times 5 \times 7 = 1260$ (omits 8)

Now we can count how many 4-digit numbers each one of the above generates, while making sure to not miscount overlaps/10-divisor numbers. This only happens at $5 \times 7 \times 8 \times 9 = 2520$, 5040, and 7560.

360 generates $25 - 3 = 22$ 4-digit, 9-divisor numbers, 840 generates 7, and 1260 generates 4, which all sums to $\boxed{33}$.

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4. A shipping company charges $.30l + .40w + .50h$ dollars to process a right rectangular prism-shaped box with dimensions l, w, h in inches. The customers themselves are allowed to label the three dimensions of their box with l, w, h for the purpose of calculating the processing fee. A customer finds that there are two different ways to label the dimensions of their box B to get a fee of \$8.10, and two different ways to label B to get a fee of \$8.70. None of the faces of B are squares. Find the surface area of B, in square inches.

Proposed by Justin Hsieh

Answer: 276

Solution. Let a, b, c be the dimensions in inches of box B. There are six possible shipment fees, in tenths of dollars:

$3a + 4b + 5c$, $3a + 4c + 5b$, $3b + 4a + 5c$, $3b + 4c + 5a$, $3c + 4a + 5b$, $3c + 4b + 5a$.

Suppose that the first two fees are equal. We get that $3a + 4b + 5c = 3a + 4c + 5b$, which simplifies to $c = b$. However, we are given that B does not contain square faces, so this case is impossible. In general, no two fees can be equal if any dimension is labeled the same way for both fees.

Suppose that the second and third fees are equal. Then we get that $3a + 4c + 5b = 3b + 4a + 5c$, which simplifies to $2b = a + c$, or $b = \frac{a+c}{2}$. In general, if two fees are equal and no dimension is labeled the same way for both fees, then we get that one dimension is the average of the other two dimensions. In other words, the dimensions of B form an arithmetic progression.

Suppose we set an arbitrary dimension, say b, to be the average of the other two, so that $b = \frac{a+c}{2}$. We get this fact from equating a fee that has $3b$ and a fee that has $5b$; there are two such pairs:

$$3a + 4c + 5b = 3b + 4a + 5c$$

and

$$3b + 4c + 5a = 3c + 4a + 5b.$$

If we substitute $b = \frac{a+c}{2}$, then the first pair is equal to $\frac{11a+13c}{2}$, and the second pair is equal to $\frac{13a+11c}{2}$. If we let $\frac{11a+13c}{2} = 81$ tenth-dollars and $\frac{13a+11c}{2} = 87$ tenth-dollars, then we get $(a, c) = (10, 4)$. We then conclude that $b = \frac{10+4}{2} = 7$.

The surface area of B is $2(10 \times 7 + 10 \times 4 + 7 \times 4) = \boxed{276}$ square inches.

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5. Alan is assigning values to lattice points on the 3d coordinate plane. First, Alan computes the roots of the cubic $20x^3 - 22x^2 + 2x + 1$ and finds that they are α, β, and γ. He finds out that each of these roots satisfy $|\alpha|, |\beta|, |\gamma| \leq 1$ On each point (x, y, z) where x, y, and z are all nonnegative integers, Alan writes down $\alpha^x\beta^y\gamma^z$. What is the value of the sum of all numbers he writes down?

Proposed by Alan Abraham

Answer: 20

Solution. The sum of all the numbers he writes down is

$$\sum_{x=0}^{\infty}\sum_{y=0}^{\infty}\sum_{z=0}^{\infty} \alpha^x\beta^y\gamma^z$$

This is equivalent to

$$= \left(\sum_{x=0}^{\infty} \alpha^x\right)\left(\sum_{y=0}^{\infty} \beta^y\right)\left(\sum_{z=0}^{\infty} \gamma^z\right)$$

Since we know that $|\alpha|, |\beta|, |\gamma| \leq 1$, each of these geometric series converge. Using the formula for the sum of an infinite geometric series, our expression is equivalent to

$$= \frac{1}{1-\alpha} \cdot \frac{1}{1-\beta} \cdot \frac{1}{1-\gamma}$$

$$= \frac{1}{(1-\alpha)(1-\beta)(1-\gamma)}$$

Consider the polynomial $P(x) = 20x^3 - 22x^2 + 2x + 1$. We know that $P(x)$ can be factored into the form $P(x) = 20(x-\alpha)(x-\beta)(x-\gamma)$. Hence, our sum is equivalent to

$$= \frac{20}{P(1)}$$

$$= \frac{20}{20 - 22 + 2 + 1}$$
$$= \boxed{20}$$

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6. Find the smallest positive integer N such that each of the 101 intervals

$$[N^2, (N+1)^2), [(N+1)^2, (N+2)^2), \cdots, [(N+100)^2, (N+101)^2)$$

contains at least one multiple of 1001.

Proposed by Kyle Lee

Answer: 485

Solution. Note that the interval between two adjacent squares $[n^2, (n+1)^2)$ has width $2n$, so if $n > 500$, we obviously contain a multiple of 1001. Now, $500^2 = 250000$, and $1001|250250$, thus in the interval for $n = 500$, our multiple of 1001 is currently 250 away from the lower bound. As we begin moving downwards from 500^2, our new lower bound's position will decrease by $500^2 - 499^2 = 999, 499^2 - 498^2 = 997, 995, 993, \ldots$, while our multiple of 1001 will decrease by 1001 every time, thus our multiple of 1001 will approach the lower bound of our interval in increments of $2, 4, 6, 8$, etc. Thus when $2 + 4 + 6 + \ldots + 2k = k(k+1)$ exceeds 250, our multiple of 1001 has surpassed the lower bound of our next interval, meaning we have skipped an interval, and cannot continue. The maximal k such that this does not occur is $k = 15$, which corresponds to $N = 500 - k = \boxed{485}$ as the smallest interval we can use.

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7. For polynomials $P(x) = a_n x^n + \cdots + a_0$, let $f(P) = a_n \cdots a_0$ be the product of the coefficients of P. The polynomials P_1, P_2, P_3, Q satisfy $P_1(x) = (x-a)(x-b)$, $P_2(x) = (x-a)(x-c)$, $P_3(x) = (x-b)(x-c)$, $Q(x) = (x-a)(x-b)(x-c)$ for some complex numbers a, b, c. Given $f(Q) = 8$, $f(P_1) + f(P_2) + f(P_3) = 10$, and $abc > 0$, find the value of $f(P_1)f(P_2)f(P_3)$.

Proposed by Justin Hsieh

Answer: 32

Solution. We compute $f(P_1) = -(a+b)ab$, $f(P_2) = -(a+c)ac$, $f(P_3) = -(b+c)bc$, and $f(Q) = (a+b+c)(ab+ac+bc)abc = 8$. Then $f(P_1)+f(P_2)+f(P_3) = -(a^2b+ab^2+a^2c+ac^2+b^2c+bc^2)$ $= -(a+b+c)(ab+ac+bc) + 3abc = 10$.

Let $\alpha = (a+b+c)(ab+ac+bc)$ and $\beta = abc$, so $\alpha\beta = 8$ and $-\alpha + 3\beta = 10$. Then $\alpha = \frac{8}{\beta}$, which we substitute into $-\alpha + 3\beta = 10$ to get $-\frac{8}{\beta} + 3\beta = 10$. This can be written as $3\beta^2 - 10\beta - 8 = (3\beta+2)(\beta-4) = 0$, so $\beta = 4$ since $\beta = abc > 0$. Then $\alpha = \frac{8}{\beta} = 2$.

We compute $f(P_1)f(P_2)f(P_3) = -(a+b)(a+c)(b+c)(abc)^2 = -(a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2 + 2abc)(abc)^2 = -(\alpha - \beta)\beta^2 = -(2-4)4^2 = \boxed{32}$.

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8. Let z be a complex number that satisfies the equation

$$\frac{z-4}{z^2-5z+1} + \frac{2z-4}{2z^2-5z+1} + \frac{z-2}{z^2-3z+1} = \frac{3}{z}.$$

Over all possible values of z, find the sum of the values of

$$\left|\frac{1}{z^2-5z+1}+\frac{1}{2z^2-5z+1}+\frac{1}{z^2-3z+1}\right|.$$

Proposed by Justin Hsieh

Answer: $\frac{11}{6}$

Solution. Multiply both sides of the given equation by z to get

$$\frac{z^2-4z}{z^2-5z+1}+\frac{2z^2-4z}{2z^2-5z+1}+\frac{z^2-2z}{z^2-3z+1}=3.$$

Then we can rewrite the fractions as

$$\begin{aligned}
& \left(1+\frac{z-1}{z^2-5z+1}\right)+\left(1+\frac{z-1}{2z^2-5z+1}\right)+\left(1+\frac{z-1}{z^2-3z+1}\right)=3 \\
\implies & 3+(z-1)\left(\frac{1}{z^2-5z+1}+\frac{1}{2z^2-5z+1}+\frac{1}{z^2-3z+1}\right)=3 \\
\implies & (z-1)\left(f(z)\right)=0,
\end{aligned}$$

letting $f:\mathbb{C}\to\mathbb{C}$ satisfy $f(z)=\dfrac{1}{z^2-5z+1}+\dfrac{1}{2z^2-5z+1}+\dfrac{1}{z^2-3z+1}$. Therefore either $z-1=0$ or $f(z)=0$. Equivalently, if $f(z)\neq 0$, then $z=1$. The value $z=1$ does indeed satisfy the original equation, and

$$f(1)=\frac{1}{-3}+\frac{1}{-2}+\frac{1}{-1}=-\frac{11}{6}.$$

The final answer is $\left|-\dfrac{11}{6}\right|+(\text{sum of 0s})=\boxed{\dfrac{11}{6}}$.