CMIMC 2022

Algebra and Number Theory Round

INSTRUCTIONS

1. Do not look at the test before the proctor starts the round.
2. This test consists of 10 short-answer problems to be solved in 60 minutes and one estimation question. Each of the short-answer questions is worth points depending on its difficulty, and the estimation question will be used to break ties. **If you do not write an estimate for estimation, you will be placed last in tiebreaking.**
3. Write your name, team name, and team ID on your answer sheet. Circle the subject of the test you are taking.
4. Write your answers in the corresponding boxes on the answer sheets.
5. No computational aids other than pencil/pen are permitted.
6. Answers must be reasonably simplified.
7. If you believe that the test contains an error, submit your protest in writing to Doherty 2302 by the end of lunch.

Algebra and Number Theory

1. Alice and Bob live on the same road. At time t, they both decide to walk to each other's houses at constant speed. However, they were busy thinking about math so that they didn't realize passing each other. Alice arrived at Bob's house at $3:19$pm, and Bob arrived at Alice's house at $3:29$pm. Charlie, who was driving by, noted that Alice and Bob passed each other at $3:11$pm. Find the difference in minutes between the time Alice and Bob left their own houses and noon on that day.

2. Arthur, Bob, and Carla each choose a three-digit number. They each multiply the digits of their own numbers. Arthur gets 64, Bob gets 35, and Carla gets 81. Then, they add corresponding digits of their numbers together. The total of the hundreds place is 24, that of the tens place is 12, and that of the ones place is 6. What is the difference between the largest and smallest of the three original numbers?

3. How many 4-digit numbers have exactly 9 divisors from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$?

4. A shipping company charges $0.30l + 0.40w + 0.50h$ dollars to process a right rectangular prism-shaped box with dimensions l, w, h in inches. The customers themselves are allowed to label the three dimensions of their box with l, w, h for the purpose of calculating the processing fee. A customer finds that there are two different ways to label the dimensions of their box B to get a fee of \$8.10, and two different ways to label B to get a fee of \$8.70. None of the faces of B are squares. Find the surface area of B, in square inches.

5. Alan is assigning values to lattice points on the 3d coordinate plane. First, Alan computes the roots of the cubic $20x^3 - 22x^2 + 2x + 1$ and finds that they are α, β, and γ. He finds out that each of these roots satisfy $|\alpha|, |\beta|, |\gamma| \leq 1$ On each point (x, y, z) where x, y, and z are all nonnegative integers, Alan writes down $\alpha^x \beta^y \gamma^z$. What is the value of the sum of all numbers he writes down?

6. Find the smallest positive integer N such that each of the 101 intervals
$$[N^2, (N+1)^2),\ [(N+1)^2, (N+2)^2),\ \cdots,\ [(N+100)^2, (N+101)^2)$$
contains at least one multiple of 1001.

7. For polynomials $P(x) = a_n x^n + \cdots + a_0$, let $f(P) = a_n \cdots a_0$ be the product of the coefficients of P. The polynomials P_1, P_2, P_3, Q satisfy $P_1(x) = (x-a)(x-b)$, $P_2(x) = (x-a)(x-c)$, $P_3(x) = (x-b)(x-c)$, $Q(x) = (x-a)(x-b)(x-c)$ for some complex numbers a, b, c. Given $f(Q) = 8$, $f(P_1) + f(P_2) + f(P_3) = 10$, and $abc > 0$, find the value of $f(P_1)f(P_2)f(P_3)$.

8. Let z be a complex number that satisfies the equation
$$\frac{z-4}{z^2-5z+1} + \frac{2z-4}{2z^2-5z+1} + \frac{z-2}{z^2-3z+1} = \frac{3}{z}.$$
Find the sum of
$$\left|\frac{1}{z^2-5z+1} + \frac{1}{2z^2-5z+1} + \frac{1}{z^2-3z+1}\right|.$$
over all possible values of z.