CMIMC 2022

Algebra and Number Theory Round

INSTRUCTIONS

1. Do not look at the test before the proctor starts the round.
2. This test consists of 10 short-answer problems to be solved in 60 minutes and one estimation question. Each of the short-answer questions is worth points depending on its difficulty, and the estimation question will be used to break ties. **If you do not write an estimate for estimation, you will be placed last in tiebreaking.**
3. Write your name, team name, and team ID on your answer sheet. Circle the subject of the test you are taking.
4. Write your answers in the corresponding boxes on the answer sheets.
5. No computational aids other than pencil/pen are permitted.
6. Answers must be reasonably simplified.
7. If you believe that the test contains an error, submit your protest in writing to Doherty 2302 by the end of lunch.

Algebra and Number Theory

1. How many 4-digit numbers have exactly 9 divisors from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$?

   Proposed by Ethan Gu

   Answer: 33

   Solution. We can quickly see that $1,\ 3,\ 4$ and 5 will be required if we want to reach 9 divisors. This also gives us $2,\ 6$, and 10. This means we can choose two from the remaining three numbers: $7,\ 8$, and 9 to reach exactly 9 divisors. This gives us:

   $8 \times 9 \times 5 = 360$ (omits 7) $8 \times 3 \times 7 \times 5 = 840$ (omits 9) $4 \times 9 \times 5 \times 7 = 1260$ (omits 8)

   Now we can count how many 4-digit numbers each one of the above generates, while making sure to not miscount overlaps/10-divisor numbers. This only happens at $5 \times 7 \times 8 \times 9 = 2520,\ 5040$, and 7560.

   360 generates $25 - 3 = 22$ 4-digit, 9-divisor numbers, 840 generates 7, and 1260 generates 4, which all sums to $\boxed{33}$.

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2. A shipping company charges $.30l + .40w + .50h$ dollars to process a right rectangular prism-shaped box with dimensions l, w, h in inches. The customers themselves are allowed to label the three dimensions of their box with l, w, h for the purpose of calculating the processing fee. A customer finds that there are two different ways to label the dimensions of their box B to get a fee of \$8.10, and two different ways to label B to get a fee of \$8.70. None of the faces of B are squares. Find the surface area of B, in square inches.

   Answer: 276

   Solution. Let a, b, c be the dimensions in inches of box B. There are six possible shipment fees, in tenths of dollars:

   $3a + 4b + 5c,\ 3a + 4c + 5b,\ 3b + 4a + 5c,\ 3b + 4c + 5a,\ 3c + 4a + 5b,\ 3c + 4b + 5a.$

   Suppose that the first two fees are equal. We get that $3a + 4b + 5c = 3a + 4c + 5b$, which simplifies to $c = b$. However, we are given that B does not contain square faces, so this case is impossible. In general, no two fees can be equal if any dimension is labeled the same way for both fees.

   Suppose that the second and third fees are equal. Then we get that $3a + 4c + 5b = 3b + 4a + 5c$, which simplifies to $2b = a + c$, or $b = \frac{a+c}{2}$. In general, if two fees are equal and no dimension is labeled the same way for both fees, then we get that one dimension is the average of the other two dimensions. In other words, the dimensions of B form an arithmetic progression.

   Suppose we set an arbitrary dimension, say b, to be the average of the other two, so that $b = \frac{a+c}{2}$. We get this fact from equating a fee that has $3b$ and a fee that has $5b$; there are two such pairs:

   $$3a + 4c + 5b = 3b + 4a + 5c$$

   and

   $$3b + 4c + 5a = 3c + 4a + 5b.$$

   If we substitute $b = \frac{a+c}{2}$, then the first pair is equal to $\frac{11a+13c}{2}$, and the second pair is equal to $\frac{13a+11c}{2}$. If we let $\frac{11a+13c}{2} = 81$ tenth-dollars and $\frac{13a+11c}{2} = 87$ tenth-dollars, then we get $(a, c) = (10, 4)$. We then conclude that $b = \frac{10+4}{2} = 7$.

The surface area of B is $2(10 \times 7 + 10 \times 4 + 7 \times 4) = \boxed{276}$ square inches.

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3. Find the smallest positive integer N such that each of the 101 intervals

$$[N^2, (N+1)^2), [(N+1)^2, (N+2)^2), \cdots, [(N+100)^2, (N+101)^2)$$

contains at least one multiple of 1001.

Proposed by Kyle Lee

Answer: 485

Solution. Note that the interval between two adjacent squares $[n^2, (n+1)^2)$ has width $2n$, so if $n > 500$, we obviously contain a multiple of 1001. Now, $500^2 = 250000$, and $1001|250250$, thus in the interval for $n = 500$, our multiple of 1001 is currently 250 away from the lower bound. As we begin moving downwards from 500^2, our new lower bound's position will decrease by $500^2 - 499^2 = 999, 499^2 - 498^2 = 997, 995, 993, \ldots$, while our multiple of 1001 will decrease by 1001 every time, thus our multiple of 1001 will approach the lower bound of our interval in increments of $2, 4, 6, 8$, etc. Thus when $2 + 4 + 6 + \ldots + 2k = k(k+1)$ exceeds 250, our multiple of 1001 has surpassed the lower bound of our next interval, meaning we have skipped an interval, and cannot continue. The maximal k such that this does not occur is $k = 15$, which corresponds to $N = 500 - k = \boxed{485}$ as the smallest interval we can use.

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4. Let z be a complex number that satisfies the equation

$$\frac{z-4}{z^2-5z+1} + \frac{2z-4}{2z^2-5z+1} + \frac{z-2}{z^2-3z+1} = \frac{3}{z}.$$

Over all possible values of z, find the sum of the values of

$$\left|\frac{1}{z^2-5z+1} + \frac{1}{2z^2-5z+1} + \frac{1}{z^2-3z+1}\right|.$$

Proposed by Justin Hsieh

Answer: $\frac{11}{6}$

Solution. Multiply both sides of the given equation by z to get

$$\frac{z^2-4z}{z^2-5z+1} + \frac{2z^2-4z}{2z^2-5z+1} + \frac{z^2-2z}{z^2-3z+1} = 3.$$

Then we can rewrite the fractions as

$$\begin{aligned}
& \left(1 + \frac{z-1}{z^2-5z+1}\right) + \left(1 + \frac{z-1}{2z^2-5z+1}\right) + \left(1 + \frac{z-1}{z^2-3z+1}\right) = 3 \\
\Longrightarrow \quad & 3 + (z-1)\left(\frac{1}{z^2-5z+1} + \frac{1}{2z^2-5z+1} + \frac{1}{z^2-3z+1}\right) = 3 \\
\Longrightarrow \quad & (z-1)\left(f(z)\right) = 0,
\end{aligned}$$

letting $f : \mathbb{C} \to \mathbb{C}$ satisfy $f(z) = \dfrac{1}{z^2 - 5z + 1} + \dfrac{1}{2z^2 - 5z + 1} + \dfrac{1}{z^2 - 3z + 1}$. Therefore either $z - 1 = 0$ or $f(z) = 0$. Equivalently, if $f(z) \neq 0$, then $z = 1$. The value $z = 1$ does indeed satisfy the original equation, and

$$f(1) = \frac{1}{-3} + \frac{1}{-2} + \frac{1}{-1} = -\frac{11}{6}.$$

The final answer is $\left|-\dfrac{11}{6}\right| + (\text{sum of 0s}) = \boxed{\dfrac{11}{6}}$.

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5. Grant is standing at the beginning of a hallway with infinitely many lockers, numbered, $1, 2, 3, \ldots$ All of the lockers are initially closed. Initially, he has some set $S = \{1, 2, 3, \ldots\}$.

   Every step, for each element s of S, Grant goes through the hallway and opens each locker divisible by s that is closed, and closes each locker divisible by s that is open. Once he does this for all s, he then replaces S with the set of labels of the currently open lockers, and then closes every door again.

   After 2022 steps, S has n integers that divide 10^{2022}. Find n.

   Proposed by Oliver Hayman

   Answer: 64

   Solution. Let's denote S_n to be the set S after n steps, so $S_0 = \{1, 2, 3, \ldots\}$. First, observe that S_{n+1} consists of all natural numbers which have an odd number of divisors from S_n. Now, we claim that all S_n are *multiplicative*, meaning that if we select two numbers a, b with $\gcd(a, b) = 1$, then $a, b \in S_n \iff ab \in S_n$. We will show this inductively; clearly S_0 is multiplicative. Suppose S_n is multiplicative, and $x = p_1^{e_1} p_2^{e_2} \ldots p_k^{e_k} \in S_{n+1}$. Now, since S_n is multiplicative, the number of divisors of x in S_n, say $d_{S_n}(x)$, is simply $d_{S_n}(p_1^{e_1}) d_{S_n}(p_2^{e_2}) \ldots d_{S_n}(p_k^{e_k})$, which is odd iff all of the component factors are, meaning $x \in S_{n+1} \iff \prod_{i \in A} p_i^{e_i} \in S_{n+1}$ for any subset $A \subseteq \{1, 2, \ldots, k\}$, which is enough to show multiplicativity.

   Now, this tells us that we can determine S_n by looking at a single prime factor, and it is also clear by symmetry that S_n acts identically on all prime factors, so we can instead consider the set E_n of possible exponents a prime factor in S_n can have. Initially $E_0 = \{0, 1, 2, \ldots\}$, $E_1 = \{0, 2, 4, \ldots\}$, and so on. By writing out the first few sets, we can observe that E_n is in fact the set of non-negative integers x such that $n \& x = 0$, where & denotes the <u>bit-wise and</u> operator. A formal proof of this, which we will omit, can be carried out by defining $f(n, x) = \delta_{n\&x} = (1 \text{ if } n\&x = 0 \text{ else } 0)$, and showing that $f(n+1, x+1) = f(n+1, x) \oplus f(n, x+1)$, where $\oplus$ denotes the <u>bit-wise xor</u> operator.

   Now, $2022 = 11111100110_2$, so elements of E_{2022} that are ≤ 2022 are just numbers of the form $b_1 b_2 00 b_3$ in base 2. There are $2^3 = 8$ of these, so we then get $8^2 = \boxed{64}$ total factors, since 10 is divisible by 2 primes, and we can choose the exponent for each independently.

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6. Find the probability such that when a polynomial in $\mathbb{Z}_{2027}[x]$ having degree at most 2026 is chosen uniformly at random,

   $$x^{2027} - x | P^k(x) - x \iff 2021 | k$$

   (note that 2027 is prime).

   Here $P^k(x)$ denotes P composed with itself k times.

Note: the problem was discounted in score calculations because the original answer was wrong.

Proposed by Grant Yu

Answer: $\dfrac{2027!}{(2027-q)!2027^{2027}q}$

Solution. Intended solution: replace 2021 by any prime q smaller than but close to $p = 2027$ (say for example 2017).

By Lagrange Interpolation it can be seen that every function $\mathbb{Z}_p[x] \to \mathbb{Z}_p[x]$ is a unique polynomial of degree $\leq p-1$. Moreover the polynomial divisibility condition can be translated as $P^k(x) = x \forall x \in \mathbb{Z}_{2027} \iff q|k$. Note that $P^q(x) = x$ implies P is in fact a permutation on $\mathbb{Z}_{2027}$ such that every cycle in the cycle decomposition of that permutation has length dividing q. Since q has only divisors $1, q$, if there are a cycles of length 1 and b cycles of length q then $a + bq = 2027$ with $b \geq 1$ (else the period would be 1 and the function would be the identity, i.e. $P(x) = x$). If q is sufficiently close to 2027, e.g. $q = 2017$, then $b \leq 1 \implies b = 1 \implies a = 2027 - q$. There are $\binom{2027}{q}$ ways to choose the fixed points and $(q-1)!$ ways to permute the q-cycle, so there are $\binom{2027}{q}(q-1)! = \frac{2027!}{(2027-q)!q}$ such functions. Since there are 2027^{2027} functions total from ${}_p$ to ${}_p$ and we are choosing one among them uniformly at random, the probability is $\frac{2027!}{(2027-q)!2027^{2027}q}$.

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7. Let $f(n)$ count the number of values $0 \leq k \leq n^2$ such that $43 \nmid \binom{n^2}{k}$. Find the least positive value of n such that
$$43^{43} \mid f\left(\frac{43^n - 1}{42}\right)$$

Proposed by Adam Bertelli

Answer: 924

Solution. To find the number of integers $0 \leq k \leq n$ satisfying $43|\binom{n}{k}$, we can use Lucas's Theorem, which tells us that $\binom{a}{b} \mod p \equiv \prod \binom{a_i}{b_i} \mod p$, where $a = (a_i a_{i-1} \dots a_0)_p$ are the (possibly zero) digits of a in base p, and likewise for b. In particular, this tells us that this number is not divisible by 43 only when every b_i is at most equal to the corresponding digit a_i (this is actually an easier result that is not hard to show inductively). Thus the number of such numbers b is precisely equal to $\prod(a_i + 1)$, since we have $a_i + 1$ possible choices for the ith digit of b to satisfy $0 \leq b_i \leq a_i$. This further implies that the number of times 43 divides this product is simply the number of digits $a_i = 42$.

Now, note that $N = \frac{43^n - 1}{42} = 11\dots1_{43}$, so $N^2 = (11\dots1)^2 = 123\dots(n-1)(n)(n-1)\dots321_{43}$, if we ignore carrying. Call this sequence of $2n-1$ digits (ignoring carrying) $c_{2n-2}, \dots, c_0$, and let the *carry level* of c_x be the number of carries added to the xth place by digits to the right of c_x, or equivalently, $\left\lfloor \dfrac{(c_{x-1} \dots c_1 c_0)_{43}}{43^{x-1}} \right\rfloor$.

Clearly the last 43 digits of N^2 have carry level 0, as they look like $0(42)(41)\dots321_{43}$. The next 42 digits from the right all have carry level 1, as each of them is followed by a digit that was originally greater than or equal to 43, and this sequence will look like $0(42)(41)\dots432_{43}$ (note that we skip the digit 1, as the digit that would've been 1 was the first one to have a higher carry level, and hence was raised by 1, along with the digits to the left of it). It is not hard to see that the next blocks of 42 digits from the right will have carry levels $2, 3, 4, \dots$, and will all look like $0(42)(41)\dots432_{43}$, and this will continue until we do not have enough increasing digits remaining to form a complete block of this form. This tells us that our maximum carry level attained is equal to $k = \left\lceil \frac{n-43}{42} \right\rceil$.

Now, if $n' = (n + k \mod 43)$, our last incomplete block will look something like

$$(42)01234\ldots(n'-1)(n')(n'-1)\ldots432_{43}$$

all with carry level k. However, as soon as we reach the 42 on the left, the next digit to the left will receive one less carry, so the next block of 42 digits will have carry level $k-1$, and look like $(42)012\ldots(38)(39)(40)_{43}$ (again we skip 41 for the same reason as before). We will again have $k-1$ total blocks of this form, and then our final block (the beginning of the number) will just be $1234\ldots(39)(40)_{43}$ with carry level 0.

Now, we can count up the number of 42 digits occurring in our blocks - there are k 42's in the back half of our number, $k-1$ in the front half, and 1 in the middle block, for a total of $2k$. However, note that it is possible for n' to equal 42, in which case we gain an extra 42 in the middle of our number. We want 43 such digits total, which is odd, hence we would like to take $n' = 42$ and $k = 21$ to get the smallest possible value of n. For k to be 21 we need $n > 42 \cdot 20 + 43 = 883$, and we also know $n \equiv n' - k \equiv 42 - 21 \mod 43$, thus the smallest possible n is $43 \cdot 21 + 21 = \boxed{924}$.

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8. Find the largest $c > 0$ such that for all $n \geq 1$ and $a_1, \ldots, a_n, b_1, \ldots, b_n > 0$ we have

$$\sum_{j=1}^{n} a_j^4 \geq c \sum_{k=1}^{n} \frac{\left(\sum_{j=1}^{k} a_j b_{k+1-j}\right)^4}{\left(\sum_{j=1}^{k} b_j^2 j!\right)^2}$$

Proposed by Grant Yu

Answer: $\frac{1}{(e-1)^2}$

Solution. We will repeatedly make use of the fact that $e - 1 = \sum_{k=1}^{\infty} \frac{1}{k!}$.

First we bound for all $k \geq 1$, $k + 1 = \binom{k+1}{k} \leq \binom{k+j}{k}$ for all $j \geq 0$ so that $\frac{1}{(k+j)!} \leq \frac{1}{(k+1)k!j!}$. This implies the bound on sum of reciprocals of factorials:

$$\frac{1}{(k+1)!} \leq \sum_{j=k+1}^{\infty} \frac{1}{j!} \leq \sum_{j=k+1}^{\infty} \frac{1}{k!} \frac{1}{(j-k)!} = \frac{1}{(k+1)k!} \sum_{t=1}^{\infty} \frac{1}{t!} = \frac{1}{k!} \cdot \frac{e-1}{k+1} \leq \frac{1}{k!} \cdot \frac{e-1}{2} < \frac{1}{k!},$$

from which we also have

$$\sum_{j=1}^{k} \frac{1}{j!} = e - 1 - \sum_{j \geq k+1} \frac{1}{j!} > e - 1 - \frac{1}{k!}.$$

Now we proceed to solve the problem.

Set $a_i = 1 \forall i$ and $b_j = \frac{1}{j!}$ for all j in the inequality, the LHS becomes n and the RHS becomes

$$\sum_{k=1}^{n} \left(\sum_{j=1}^{k} \frac{1}{j!}\right)^2 \geq \sum_{k=1}^{n} (e - 1 - \frac{1}{k!})^2.$$

So for the inequality to hold it must be the case that $nc \geq \sum_{k=1}^{n}(e-1-\frac{1}{k!})^2$ for all n, i.e. $c \geq \sup_{n\in\mathbb{N}} \frac{1}{n}\sum_{k=1}^{n}(e-1-\frac{1}{k!})^2$. Since $(e-1-\frac{1}{k!})_{k\geq 1}$ converges to $e-1$ and is strictly increasing, for any $\varepsilon > 0$ there exists N such that $(e-1-\frac{1}{k!})^2 \geq (e-1)^2-\varepsilon$ for all $k > N$, hence $c \geq \sup_{n\geq 1} \frac{1}{n}\sum_{k=1}^{n}(e-1-\frac{1}{k!})^2 \geq (e-1)^2-\varepsilon$ for all $\varepsilon > 0$, which means that

$$c \leq \frac{1}{(e-1)^2}.$$

We can use Hölder to show $c = \frac{1}{(e-1)^2}$ is optimal: note that

$$\begin{aligned}
&(e-1)\sum_{j=1}^{k}\frac{a_j^4}{(k+1-j)!}\left(\sum_{j=1}^{k} b_{k+1-j}^2(k+1-j)!\right)^2 \\
\geq &\sum_{j=1}^{k}\frac{a_j^4}{(k+1-j)!}\sum_{j=1}^{k}\frac{1}{(k+1-j)!}\left(\sum_{j=1}^{k} b_{k+1-j}^2(k+1-j)!\right)^2 \\
\geq &\left(\sum_{j=1}^{k} a_j b_{k+1-j}\right)^4.
\end{aligned}$$

Summing over all k we have

$$(e-1)\sum_{k=1}^{n}\sum_{j=1}^{k}\frac{a_j^4}{(k+1-j)!} \geq \sum_{k=1}^{n}\frac{\left(\sum_{j=1}^{k} a_j b_{k+1-j}\right)^4}{\left(\sum_{j=1}^{k} b_{k+1-j}^2(k+1-j)!\right)^2} = RHS,$$

and we may swap sums on the left double sum to obtain $\sum_{k=1}^{n}\sum_{j=1}^{k}\frac{a_j^4}{(k+1-j)!} = \sum_{k=1}^{n}\sum_{j=1}^{k}\frac{a_k^4}{j!} \leq \sum_{k=1}^{n}(e-1)a_k^4$, so when $c = \frac{1}{(e-1)^2}$, we have $\frac{LHS}{c} = (e-1)^2\sum_{k=1}^{n} a_k^4 \geq RHS$ of the original inequality, so we are done.