CMIMC 2022
Algebra and Number Theory Round

INSTRUCTIONS

1. Do not look at the test before the proctor starts the round.
2. This test consists of 10 short-answer problems to be solved in 60 minutes and one estimation question. Each of the short-answer questions is worth points depending on its difficulty, and the estimation question will be used to break ties. **If you do not write an estimate for estimation, you will be placed last in tiebreaking.**
3. Write your name, team name, and team ID on your answer sheet. Circle the subject of the test you are taking.
4. Write your answers in the corresponding boxes on the answer sheets.
5. No computational aids other than pencil/pen are permitted.
6. Answers must be reasonably simplified.
7. If you believe that the test contains an error, submit your protest in writing to Doherty 2302 by the end of lunch.

Algebra and Number Theory

1. How many 4-digit numbers have exactly 9 divisors from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$?

2. A shipping company charges $0.30l + 0.40w + 0.50h$ dollars to process a right rectangular prism-shaped box with dimensions l, w, h in inches. The customers themselves are allowed to label the three dimensions of their box with l, w, h for the purpose of calculating the processing fee. A customer finds that there are two different ways to label the dimensions of their box B to get a fee of \$8.10, and two different ways to label B to get a fee of \$8.70. None of the faces of B are squares. Find the surface area of B, in square inches.

3. Find the smallest positive integer N such that each of the 101 intervals
$$[N^2, (N+1)^2),\ [(N+1)^2, (N+2)^2),\ \cdots,\ [(N+100)^2, (N+101)^2)$$
contains at least one multiple of 1001.

4. Let z be a complex number that satisfies the equation
$$\frac{z-4}{z^2-5z+1} + \frac{2z-4}{2z^2-5z+1} + \frac{z-2}{z^2-3z+1} = \frac{3}{z}.$$
Find the sum of
$$\left|\frac{1}{z^2-5z+1} + \frac{1}{2z^2-5z+1} + \frac{1}{z^2-3z+1}\right|.$$
over all possible values of z.

5. Grant is standing at the beginning of a hallway with infinitely many lockers, numbered, $1, 2, 3, \ldots$ All of the lockers are initially closed. Initially, he has some set $S = \{1, 2, 3, \ldots\}$.

   Every step, for each element s of S, Grant goes through the hallway and opens each locker divisible by s that is closed, and closes each locker divisible by s that is open. Once he does this for all s, he then replaces S with the set of labels of the currently open lockers, and then closes every door again.

   After 2022 steps, S has n integers that divide 10^{2022}. Find n.

6. Find the probability such that when a polynomial in $\mathbb{Z}_{2027}[x]$ having degree at most 2026 is chosen uniformly at random,
$$x^{2027} - x | P^k(x) - x \iff 2021|k$$
(note that 2027 is prime).

   Here $P^k(x)$ denotes P composed with itself k times.

7. Let $f(n)$ count the number of values $0 \le k \le n^2$ such that $43 \nmid \binom{n^2}{k}$. Find the least positive value of n such that
$$43^{43} \mid f\left(\frac{43^n - 1}{42}\right)$$

8. Find the largest $c > 0$ such that for all $n \ge 1$ and $a_1, \ldots, a_n, b_1, \ldots, b_n > 0$ we have
$$\sum_{j=1}^{n} a_j^4 \ge c \sum_{k=1}^{n} \frac{\left(\sum_{j=1}^{k} a_j b_{k+1-j}\right)^4}{\left(\sum_{j=1}^{k} b_j^2 j!\right)^2}$$