CMIMC 2021

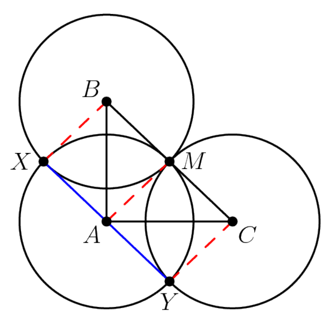
Geometry Div. 2 Solutions

1. Triangle ABC has a right angle at A, $AB = 20$, and $AC = 21$. Circles ω_A, ω_B, and ω_C are centered at A, B, and C respectively and pass through the midpoint M of $\overline{BC}$. ω_A and ω_B intersect at $X \neq M$, and ω_A and ω_C intersect at $Y \neq M$. Find XY.

   Proposed by Connor Gordon

   Answer: 29

   Solution:



   First, note that $BC = \sqrt{20^2 + 21^2} = 29$ and $AM = BM = CM$ by properties of right triangles.

   By symmetry, X and Y are the reflections of M over $\overline{AB}$ and $\overline{BC}$ respectively. Since reflections preserve distances, $XB = XA = BM = AM$, so $AMBX$ is a rhombus. Similarly, $AMCY$ is a rhombus with the same side length. It follows that $XBCY$ is a parallelogram, so $XY = BC = \boxed{29}$.

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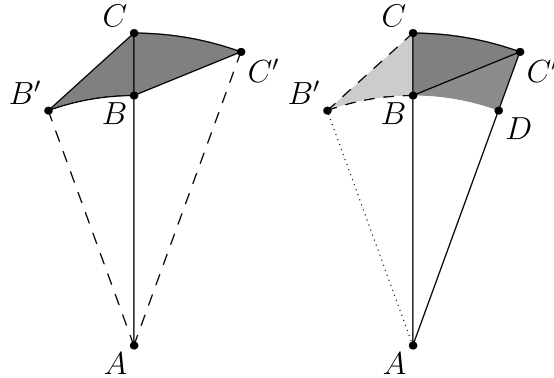
2. Points A, B, and C lie on a line, in that order, with $AB = 8$ and $BC = 2$. B is rotated 20° counterclockwise about A to a point B', tracing out an arc R_1. C is then rotated 20° clockwise about A to a point C', tracing out an arc R_2. What is the area of the region bounded by arc R_1, segment $B'C$, arc R_2, and segment $C'B$?

   Proposed by Thomas Lam

   Answer: 2π

   Solution:



Move the entire region enclosed by $\triangle AB'C$ via a $20°$ rotation clockwise about A, so that it lies on top of $\triangle ABC'$, as shown above. Let the image of B under the rotation be D. Then we see that the new area we need to find is the one bounded by arc BD, segment DC', arc CC', and segment BC. But this is just the difference between the areas of two $20°$ sectors, which have areas $\frac{1}{18} \cdot 8^2\pi$ and $\frac{1}{18} \cdot 10^2\pi$. Hence the desired area is $\frac{\pi}{18}(10^2 - 8^2) = \boxed{2\pi}$.

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3. Consider trapezoid $[ABCD]$ which has $AB \parallel CD$ with $AB = 5$ and $CD = 9$. Moreover, $\angle C = 15°$ and $\angle D = 75°$. Let M_1 be the midpoint of AB and M_2 be the midpoint of CD. What is the distance M_1M_2?

Proposed by Daniel Li

Answer: 2

Solution: Extend sides DA and CB so that they meet at a common point E. Note that $\triangle DEC$ is a right triangle with $\angle DEC = 90°$. Thus, take the circumcircles of $\triangle AEB$ and $\triangle DEC$ and note that AB and DC are the diameters of the respective circumcircles. In particular, M_1 and M_2 are the centers of the two circles. Thus, $EM_1 = \frac{5}{2}$ and $EM_2 = \frac{9}{2}$, which means that $M_1M_2 = EM_2 - EM_1 = 2$.

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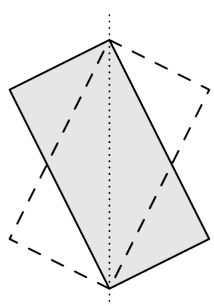
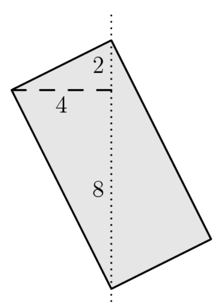
4. A $2\sqrt{5}$ by $4\sqrt{5}$ rectangle is rotated by an angle θ about one of its diagonals. If the total volume swept out by the rotating rectangle is 62π, find the measure of θ in degrees.

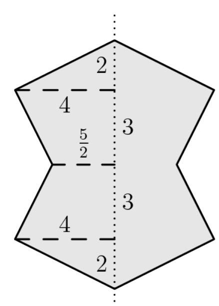
Proposed by Connor Gordon

Answer: 228

Solution: Consider the following three diagrams, where the labeled lengths can be obtained with some simple similar triangle geometry.







We use radians for arithmetic convenience. Consider the plane of the rectangle as it is revolving. For $\theta < \pi$, the cross section of the solid taken through this plane looks like the second diagram. This cross section sweeps out four "portions of cones." Two of these cones have radius 4 and height 2, and the other two have radius 4 and height 8. This gives a volume

$$V(\theta) = \frac{\theta}{2\pi}\left(2 \cdot \frac{\pi}{3} \cdot 4^2 \cdot 2 + 2 \cdot \frac{\pi}{3} \cdot 4^2 \cdot 8\right) = \frac{160\theta}{3}$$

For $0 \le \theta < \pi$. Plugging in $\theta = \pi$ gives $V = \frac{160\pi}{3} < 62\pi$, so $\theta > \pi$.

For $\theta > \pi$, the rectangle starts to overlap with existing swept-out volume, creating a cross section that looks like the third diagram. This cross section sweeps out four "portions of cones" and four "portions of frustums." The four cones all have radius 4 and height 2, and the four frustums all can be viewed as the difference between a cone of radius 4 and height 8 and a cone of radius $\frac{5}{2}$ and height 5. Considering just the volume with this cross section, we get

$$V'(\theta) = \frac{\theta - \pi}{2\pi}\left(4 \cdot \frac{\pi}{3} \cdot 4^2 \cdot 2 + 4 \cdot \frac{\pi}{3} \cdot \left[4^2 \cdot 8 - \left(\frac{5}{2}\right)^2 \cdot 5\right]\right) = \frac{515(\theta - \pi)}{6}$$

However, this volume ends up "overriding" part of our previous swept-out region, so to get the full volume we take

$$V(\theta) = \frac{160\pi}{3} + (\theta - \pi)\left(\frac{515}{6} - \frac{160}{3}\right) = \frac{195\theta}{6} + \frac{125\pi}{6}.$$

Setting this equal to 62π gives $\theta = \frac{19\pi}{15}$, which is equivalent to $\boxed{228}$ degrees.

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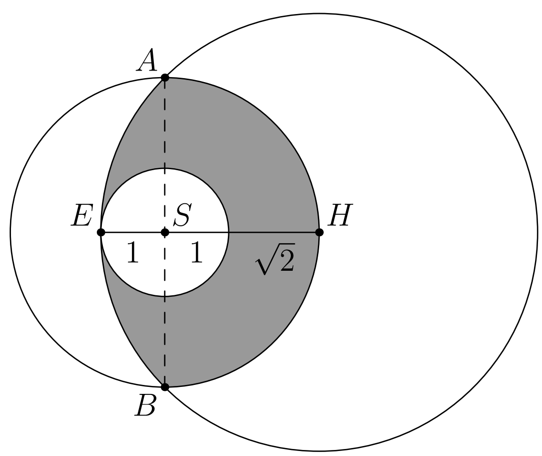
5. Emily is at $(0,0)$, chilling, when she sees a spider located at $(1,0)$! Emily runs a continuous path to her home, located at $(\sqrt{2}+2, 0)$, such that she is always moving away from the spider and toward her home. That is, her distance from the spider always increases whereas her distance to her home always decreases. What is the area of the set of all points that Emily could have visited on her run home?

Proposed by Thomas Lam

Answer: $(2+2\sqrt{2})\pi - 3 - 2\sqrt{2}$

Solution:



Let E, S, H be Emily's starting location $(0,0)$, the spider's location $(1,0)$, and Emily's house's location $(2+\sqrt{2}, 0)$, respectively. We make the following observations about the nature of Emily's path:

- Since Emily starts 1 away from the spider, she can never be less than 1 away from the spider, so she will never enter the circle centered at S with radius 1.
- Since Emily starts $2+\sqrt{2}$ away from her house, she can never be more than $2+\sqrt{2}$ away from her house, so she always stays within the circle centered at H with radius $2+\sqrt{2}$.
- Since Emily's path ends at H, where she will be $1+\sqrt{2}$ away from the spider, she never could have been more than $1+\sqrt{2}$ away from the spider (or else at some point, her distance to the spider must decrease toward $1+\sqrt{2}$), so she always stays within the circle centered at S with radius $1+\sqrt{2}$.

This narrows down the possible points that Emily can visit to the shaded region shown above. In fact, this is exhaustive. To see this, mark the intersections A and B as shown in the diagram. For every point P in the shaded region, we will describe a path that Emily can take that passes through P.

Consider two cases: Either P lies to the left of $\overline{AB}$ or to the right of $\overline{AB}$. In the first case, we claim that Emily can visit P by considering a point O on segment $\overline{SH}$ such that the circle with center O and radius OE will pass through P. Then Emily can follow an arc along this circle to reach P. Then she may finish her path by walking straight to the right until she hits the boundary of the shaded region, where she can then follow the boundary until she reaches H.

In the second case, where P lies in the subregion of the shaded region to the right of $\overline{AB}$, Emily first draws a ray from P that points left, and marks the intersection Q of this ray with the boundary of the subregion. If Q lies on the circle with radius 1 centered at S, she can reach Q by following an arc along the circumference of this circle. Otherwise Q lies along some segment of the boundary contained by $\overline{AB}$, which means Q should be reachable by following an arc as described in the first case. Either way, Emily has a path to Q. Now she walks straight to the right, passing through P, all the way to the boundary of the region, and follows the boundary to her home. We leave it as an exercise to show that the paths that we described are valid. (Note: If we require Emily's distances from the spider and her house to be *strictly* increasing and decreasing respectively, these paths don't technically work, but they can be adjusted to account for this.)

To compute the shaded area, observe that $AS \perp EH$. Thus, we need to first find the area of the semicircle with diameter $\overline{AB}$ (which is just $\frac{1}{2}\pi(1+\sqrt{2})^2$), add the area of the portion of the circle centered at H with radius $2+\sqrt{2}$ that lies to the left of $\overline{AB}$ (this can be found by finding the area of the quarter circle bounded by HA and HB and subtracting $[ABH]$, which gives $\frac{1}{4}\pi(2+\sqrt{2})^2 - \frac{1}{2}(2+\sqrt{2})^2$), and then finally subtracting the area of the unit circle centered at S (which has area π). Simplifying, this comes out to $\boxed{(2+2\sqrt{2})\pi - 3 - 2\sqrt{2}}$.

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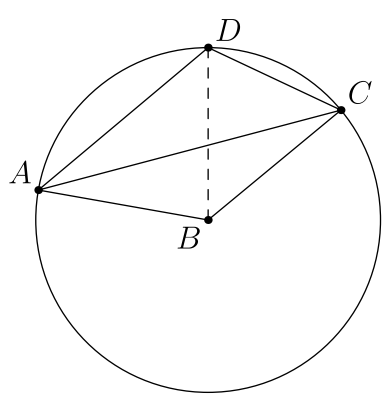
6. In convex quadrilateral $ABCD$, $\angle ADC = 90^\circ + \angle BAC$. Given that $AB = BC = 17$, and $CD = 16$, what is the maximum possible area of the quadrilateral?

Proposed by Thomas Lam

Answer: $\dfrac{529}{2}$

Solution:



Let $\angle BAC = \theta$, so that $\angle ADC = 90^\circ + \theta$. Consider the circle Ω centered at B with radius 17. We know that A and C lie on Ω. Since $\angle ABC = 180^\circ - 2\theta$ is a central angle, it follows that the major arc AC has measure $180^\circ - 2\theta$. Hence, any inscribed angle $AD'C$ with D' on minor arc AC will have measure $90^\circ - \theta$. But $\angle ADC = 90^\circ - \theta$, so D lies on Ω.

Therefore, we have $BD = 17$. This makes $\triangle BCD$ a triangle with side lengths $16, 17, 17$. Dropping an altitude from B, it is easy to compute $[BCD] = 120$. Now we only need to maximize $[ABD]$. Note that lengths AB and BD are fixed whereas $\angle ABD$ may vary. Thus $[ABD]$ is maximized when $\angle ABD$

is right, giving a maximum area of $\frac{1}{2} \cdot 17^2 = \frac{289}{2}$. Hence the maximum possible area for quadrilateral $ABCD$ is $120 + \frac{289}{2} = \boxed{\dfrac{529}{2}}$.

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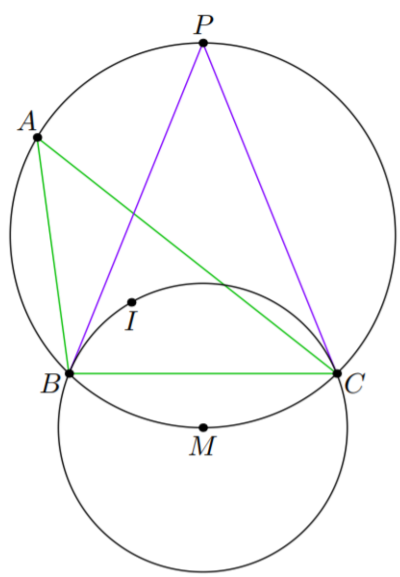
7. Let $\triangle ABC$ be a triangle with $AB = 10$ and $AC = 16$, and let I be the intersection of the internal angle bisectors of $\triangle ABC$. Suppose the tangents to the circumcircle of $\triangle BIC$ at B and C intersect at a point P with $PA = 8$. Compute the length of BC.

Proposed by Kyle Lee

Answer: $3\sqrt{14}$



Solution: Let line AI intersect $\widehat{BC}$ at point M. It is well-known that M is the midpoint of $\widehat{BC}$ by Fact 5. Now,

$$\angle BAM = \angle CAM = \angle MBC = \angle MCB = \angle BPM = \angle CPM,$$

so P lies on the circumcircle of $\triangle ABC$. By Ptolemy's Theorem, $10 \cdot PC + 8 \cdot BC = 16 \cdot PB$. Since $PB = PC$, this simplifies to $4 \cdot BC = 3 \cdot PC$. Therefore, $\cos \angle PBC = \frac{3}{8}$. By the Law of Cosines on $\triangle PAC$,

$$PC^2 = 8^2 + 16^2 - 2(8)(16)(\tfrac{3}{8}) = 224.$$

Then, $BC = (\frac{3}{4}) \cdot PC = (\frac{3}{4}) \cdot \sqrt{224} = \boxed{3\sqrt{14}}$.

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8. Let $ABCDEF$ be an equilateral hexagon such that $\triangle ACE \cong \triangle DFB$. Given that $AC = 7$, $CE = 8$, and $EA = 9$, what is the side length of this hexagon?

Proposed by Thomas Lam

Answer: $\dfrac{21\sqrt{5}}{10}$

Solution: Since the hexagon is equilateral and the triangles are congruent, $AFDC$ is a parallelogram, so $AF||CD$. It follows that $\angle AFB + \angle CDB = \angle FBD$. But $\angle AFB = \angle ABF$ and $\angle CDB = \angle CBD$. Hence, $\angle ABF + \angle FBD + \angle CBD = 2\angle FBD$, so $\angle ABC = 2\angle FBD = 2\angle AEC$. Let O be the circumcenter of $\triangle ACE$, so now $\angle ABC = \angle AOC$. Now we have that $ABCO$ is a rhombus, so the side length of the hexagon is the circumradius of $\triangle ACE$. Applying Heron's and $R = \frac{abc}{4K}$, we get an answer of $\boxed{\dfrac{21\sqrt{5}}{10}}$.

Alternate Solution:

Let O and H be the circumcenter and orthocenter of $\triangle ACE$, respectively. Observe that O is the orthocenter of $\triangle DFB$ and H is the circumcenter of $\triangle DFB$. From this symmetry, it follows that $BOEH$ is a parallelogram.

Now recall that $\overrightarrow{OA} + \overrightarrow{OC} + \overrightarrow{OE} = \overrightarrow{OH}$. Note that $\overrightarrow{OH} - \overrightarrow{OE} = \overrightarrow{OH} + \overrightarrow{EO} = \overrightarrow{OH} + \overrightarrow{HB} = \overrightarrow{OB}$, hence $\overrightarrow{OA} + \overrightarrow{OC} = \overrightarrow{OH} - \overrightarrow{OE} = \overrightarrow{OB}$. This rearranges to $\overrightarrow{OC} = \overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{AB}$. Thus $OC = AB$. But OC is the circumradius of $\triangle ACE$, thus the side length of the hexagon is precisely this circumradius. Applying Heron's and $R = \frac{abc}{4K}$, we get an answer of $\boxed{\dfrac{21\sqrt{5}}{10}}$.