CMIMC 2021

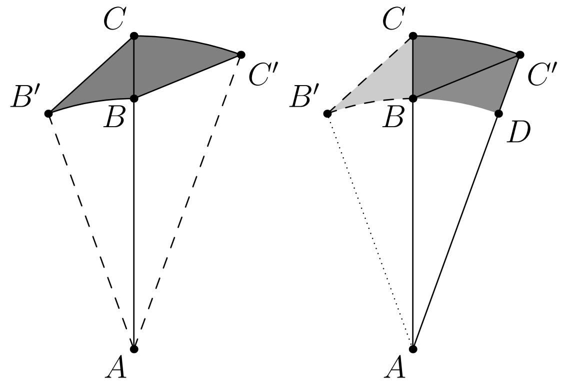
Geometry Div. 1 Solutions

1. Points A, B, and C lie on a line, in that order, with $AB = 8$ and $BC = 2$. B is rotated 20° counterclockwise about A to a point B', tracing out an arc R_1. C is then rotated 20° clockwise about A to a point C', tracing out an arc R_2. What is the area of the region bounded by arc R_1, segment $B'C$, arc R_2, and segment $C'B$?

   Proposed by Thomas Lam

   Answer: 2π

   Solution:



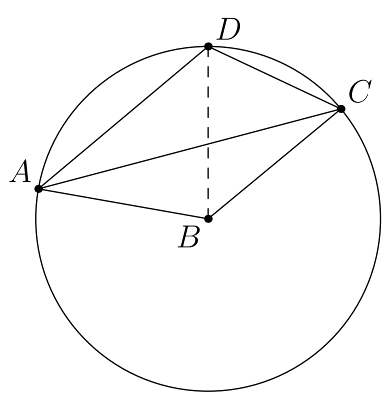
   Move the entire region enclosed by $\triangle AB'C$ via a 20° rotation clockwise about A, so that it lies on top of $\triangle ABC'$, as shown above. Let the image of B under the rotation be D. Then we see that the new area we need to find is the one bounded by arc BD, segment DC', arc CC', and segment BC. But this is just the difference between the areas of two 20° sectors, which have areas $\frac{1}{18} \cdot 8^2\pi$ and $\frac{1}{18} \cdot 10^2\pi$. Hence the desired area is $\frac{\pi}{18}(10^2 - 8^2) = \boxed{2\pi}$.

2. In convex quadrilateral $ABCD$, $\angle ADC = 90^\circ + \angle BAC$. Given that $AB = BC = 17$, and $CD = 16$, what is the maximum possible area of the quadrilateral?

   Proposed by Thomas Lam

   Answer: $\dfrac{529}{2}$

   Solution:



Let $\angle BAC = \theta$, so that $\angle ADC = 90^\circ + \theta$. Consider the circle Ω centered at B with radius 17. We know that A and C lie on Ω. Since $\angle ABC = 180^\circ - 2\theta$ is a central angle, it follows that the major arc AC has measure $180^\circ - 2\theta$. Hence, any inscribed angle $AD'C$ with D' on minor arc AC will have measure $90^\circ - \theta$. But $\angle ADC = 90^\circ - \theta$, so D lies on Ω.

Therefore, we have $BD = 17$. This makes $\triangle BCD$ a triangle with side lengths $16, 17, 17$. Dropping an altitude from B, it is easy to compute $[BCD] = 120$. Now we only need to maximize $[ABD]$. Note that lengths AB and BD are fixed whereas $\angle ABD$ may vary. Thus $[ABD]$ is maximized when $\angle ABD$ is right, giving a maximum area of $\frac{1}{2} \cdot 17^2 = \frac{289}{2}$. Hence the maximum possible area for quadrilateral $ABCD$ is $120 + \frac{289}{2} = \boxed{\frac{529}{2}}$.

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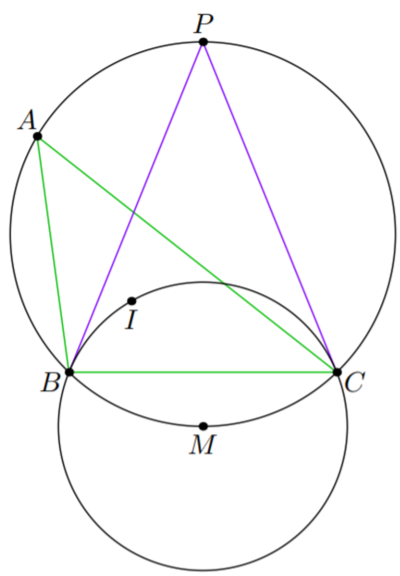
3. Let $\triangle ABC$ be a triangle with $AB = 10$ and $AC = 16$, and let I be the intersection of the internal angle bisectors of $\triangle ABC$. Suppose the tangents to the circumcircle of $\triangle BIC$ at B and C intersect at a point P with $PA = 8$. Compute the length of BC.

Proposed by Kyle Lee

Answer: $3\sqrt{14}$



Solution: Let line AI intersect $\widehat{BC}$ at point M. It is well-known that M is the midpoint of $\widehat{BC}$ by Fact 5. Now,

$$\angle BAM = \angle CAM = \angle MBC = \angle MCB = \angle BPM = \angle CPM,$$

so P lies on the circumcircle of $\triangle ABC$. By Ptolemy's Theorem, $10 \cdot PC + 8 \cdot BC = 16 \cdot PB$. Since $PB = PC$, this simplifies to $4 \cdot BC = 3 \cdot PC$. Therefore, $\cos \angle PBC = \frac{3}{8}$. By the Law of Cosines on $\triangle PAC$,

$$PC^2 = 8^2 + 16^2 - 2(8)(16)(\tfrac{3}{8}) = 224.$$

Then, $BC = (\frac{3}{4}) \cdot PC = (\frac{3}{4}) \cdot \sqrt{224} = \boxed{3\sqrt{14}}$.

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4. Let $ABCDEF$ be an equilateral heaxagon such that $\triangle ACE \cong \triangle DFB$. Given that $AC = 7$, $CE = 8$, and $EA = 9$, what is the side length of this hexagon?

Proposed by Thomas Lam

Answer: $\dfrac{21\sqrt{5}}{10}$

Solution: Since the hexagon is equilateral and the triangles are congruent, $AFDC$ is a parallelogram, so $AF||CD$. It follows that $\angle AFB + \angle CDB = \angle FBD$. But $\angle AFB = \angle ABF$ and $\angle CDB = \angle CBD$. Hence, $\angle ABF + \angle FBD + \angle CBD = 2\angle FBD$, so $\angle ABC = 2\angle FBD = 2\angle AEC$. Let O be the circumcenter of $\triangle ACE$, so now $\angle ABC = \angle AOC$. Now we have that $ABCO$ is a rhombus, so the side length of the hexagon is the circumradius of $\triangle ACE$. Applying Heron's and $R = \frac{abc}{4K}$, we get an answer of $\boxed{\dfrac{21\sqrt{5}}{10}}$.

Alternate Solution:

Let O and H be the circumcenter and orthocenter of $\triangle ACE$, respectively. Observe that O is the orthocenter of $\triangle DFB$ and H is the circumcenter of $\triangle DFB$. From this symmetry, it follows that $BOEH$ is a parallelogram.

Now recall that $\overrightarrow{OA} + \overrightarrow{OC} + \overrightarrow{OE} = \overrightarrow{OH}$. Note that $\overrightarrow{OH} - \overrightarrow{OE} = \overrightarrow{OH} + \overrightarrow{EO} = \overrightarrow{OH} + \overrightarrow{HB} = \overrightarrow{OB}$, hence $\overrightarrow{OA} + \overrightarrow{OC} = \overrightarrow{OH} - \overrightarrow{OE} = \overrightarrow{OB}$. This rearranges to $\overrightarrow{OC} = \overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{AB}$. Thus $OC = AB$. But OC is the circumradius of $\triangle ACE$, thus the side length of the hexagon is precisely this circumradius. Applying Heron's and $R = \frac{abc}{4K}$, we get an answer of $\boxed{\dfrac{21\sqrt{5}}{10}}$.

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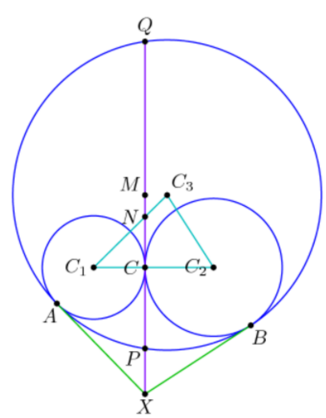
5. Let $\gamma_1, \gamma_2, \gamma_3$ be three circles with radii $3, 4, 9$, respectively, such that γ_1 and γ_2 are externally tangent at C, and γ_3 is internally tangent to γ_1 and γ_2 at A and B, respectively. Suppose the tangents to γ_3 at A and B intersect at X. The line through X and C intersect γ_3 at two points, P and Q. Compute PQ.

Proposed by Kyle Lee

Answer: $\dfrac{72\sqrt{3}}{7}$

Solution:



Let C_n denote the center of γ_n for $n = 1, 2, 3$, and let M denote the midpoint of PQ. It is well-known that $C_3M \perp PQ$. If $N = C_1C_3 \cap CM$, then we have $\triangle CC_1N \sim C_3MN$. Now, observe that since

$C_2C_3 = 9-4 = 5, C_3C_1 = 9-3 = 6$, and $C_1C_2 = 3+4 = 7$, we have that $\triangle C_1C_2C_3$ is a $5-6-7$ triangle. Then by the Law of Cosines, $\cos\angle C_2C_1C_3 = \frac{5}{7}$, so $C_1N = 3\cdot\frac{7}{5} = \frac{21}{5}$ and $C_3N = 6 - \frac{21}{5} = \frac{9}{5}$. Finally, by the similar triangles, we have $C_3M = \frac{9}{5}\cdot\frac{5}{7} = \frac{9}{7}$, so $MP = MQ = \sqrt{9^2 - (\frac{9}{7})^2} = \frac{36\sqrt{3}}{7}$ and $PQ = \boxed{\frac{72\sqrt{3}}{7}}$.

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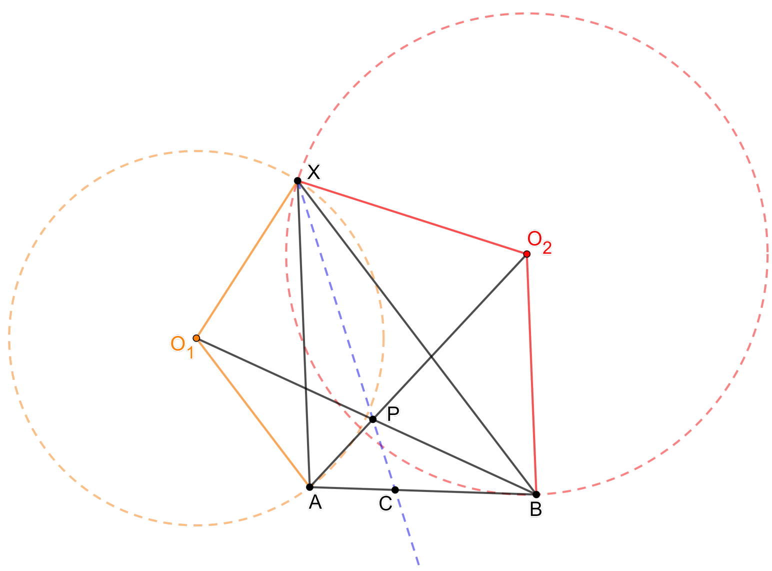
6. Let circles ω and Γ, centered at O_1 and O_2 and having radii 42 and 54 respectively, intersect at points X, Y, such that $\angle O_1XO_2 = 105^\circ$. Points A, B lie on ω and Γ respectively such that $\angle O_1XA = \angle AXB = \angle BXO_2$ and Y lies on both minor arcs XA and XB. Define P to be the intersection of AO_2 and BO_1. Suppose XP intersects AB at C. What is the value of $\frac{AC}{BC}$?

   Proposed by Puhua Cheng

   Answer: $\dfrac{49}{81}$

   Solution: The problem can be solved with the trigonometric form of Ceva's theorem and repeated applications of sine rule.

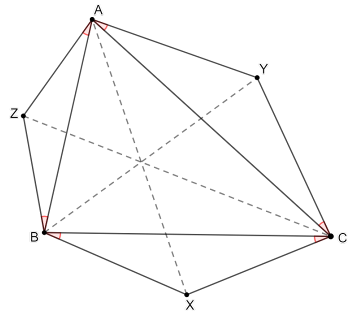


$$\begin{aligned}
\frac{\sin\angle AXC}{\sin\angle BXC} &= \frac{\sin\angle XAO_2}{\sin\angle BAO_2}\cdot\frac{\sin\angle ABO_1}{\sin\angle XBO_1}\\
&= \frac{\sin\angle XAO_2/XO_2}{\sin\angle BAO_2/BO_2}\cdot\frac{\sin\angle ABO_1/AO_1}{\sin\angle XBO_1/XO_1}\\
&= \frac{\sin\angle AXO_2/AO_2}{\sin\angle ABO_2/AO_2}\cdot\frac{\sin\angle BAO_1/BO_1}{\sin\angle BXO_1/BO_1}\\
&= \frac{\sin\angle BAO_1}{\sin\angle ABO_2} = \frac{\sin\angle XBA}{\sin\angle XAB} = \frac{XA}{XB}
\end{aligned}$$

$$\frac{AC}{BC} = \frac{AC/\sin\angle AXC}{BC/\sin\angle BXC}\cdot\frac{\sin\angle AXC}{\sin\angle BXC} = \frac{XA/\sin\angle XCA}{XB/\sin\angle XCB}\cdot\frac{XA}{XB} = \left(\frac{XA}{XB}\right)^2 = \left(\frac{O_1X}{O_2X}\right)^2 = \boxed{\frac{49}{81}}$$

Comment: It is worth noting that XP is the symmedian of $\triangle XAB$. This is clear with the help of the following lemma:

Lemma. Given a triangle $\triangle ABC$, construct similar isosceles triangles $\triangle XBC \sim \triangle YCA \sim \triangle ZAB$ with $XB = XC$. Then AX, BY, CZ are concurrent.



The lemma can be proven similarly with the trigonometric form of Ceva's theorem. A generalized version is known as Jacobi's theorem.

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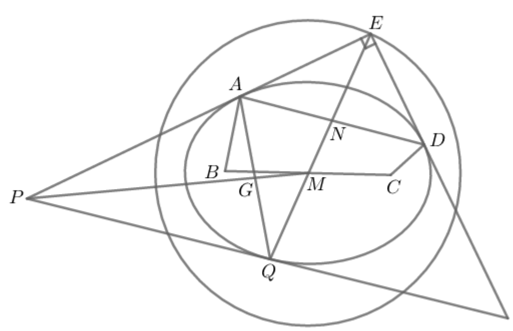
7. Convex pentagon $ABCDE$ has $\overline{BC} = 17$, $\overline{AB} = 2\overline{CD}$, and $\angle E = 90°$. Additionally, $\overline{BD} - \overline{CD} = \overline{AC}$, and $\overline{BD} + \overline{CD} = 25$. Let M and N be the midpoints of BC and AD respectively. Ray EA is extended out to point P, and a line parallel to AD is drawn through P, intersecting line EM at Q. Let G be the midpoint of AQ. Given that N and G lie on EM and PM respectively, and the perimeter of $\triangle QBC$ is 42, find the length of $\overline{EM}$.

   Proposed by Adam Bertelli

   Answer: $\dfrac{31}{2}$

   Solution:



We can first observe that the stated conditions imply $\overline{BD} + \overline{CD} = \overline{AB} + \overline{AC} = \overline{QB} + \overline{QC} = 25$, so A, D, Q lie on ellipse ε with foci B, C. Now, there exists an affine transform sending this ellipse to a circle, and affine transforms preserve collinearity and ratios between lengths on the same line, so consider the diagram under this transform, and let any point X be sent to X'. Clearly M is sent to the center of the circle by symmetry, thus we have that points A', D' lie on circle ε', and their midpoint lies on $E'M'$, thus $E'A' = E'D'$. Since $P'Q' || A'D'$, and Q' lies on $E'M'$, this also tells us that $P'Q'$ is tangent to ε' at Q'. Finally, the line from M' through G' should meet the intersection point of the tangents to ε' at A', Q', and since this line passes through P', it follows that $P'A'$, and by extension $A'E'$, is tangent to ε', and by symmetry $D'E'$ is tangent as well.

Thus we now know that, in the original configuration, EA, ED are tangents to the ellipse, so E being a right angle tells us that E lies on the *director circle* of ε, meaning $EM = a^2 + b^2$, where a, b are the lengths of the semi-major and semi-minor axes. Clearly $a = \frac{25}{2}$, and $b = \frac{1}{2}\sqrt{25^2 - 17^2}$ by the Pythagorean theorem, so $EM = \sqrt{a^2 + b^2} = \frac{1}{2}\sqrt{25^2 + 25^2 - 17^2} = \boxed{\frac{31}{2}}$.

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8. Let ABC be a triangle with $AB < AC$ and ω be a circle through A tangent to both the B-excircle and the C-excircle. Let ω intersect lines AB, AC at X, Y respectively and X, Y lie outside of segments AB, AC. Let O be the center of ω and let OI_C, OI_B intersect line BC at J, K respectively. Suppose $KJ = 4$, $KO = 16$ and $OJ = 13$. Find $\frac{[KI_BI_C]}{[JI_BI_C]}$.

   Proposed by Grant Yu

   Answer: $\frac{103}{71}$

   Solution: The key observation is that the inversion centered at A swapping $(I_C), (I_B)$ also swaps ω and line BC. From here we deduce that $Q := OA \cap BC$ satisfies $AQ \perp BC$.

   Introduce D, E as the points of contact of $(I_C), (I_B)$ with BC. Let F, G be the points of contact of ω with $(I_B), (I_C)$ respectively. By Monge on $\omega, (I_C), (I_B)$ it follows that I_BI_C, GF, BC concur at, say, H.

   We have $(HQ; KJ) \stackrel{O}{=} (HA; I_BI_C) = -1$. Afterwards the problem pretty much reduces to working with the right triangle OQH and points J, K with $(KJ; HQ) = -1$. Let $x = JQ$ we have $13^2 - x^2 = 16^2 - (4 + x)^2 \implies x = \frac{71}{8}$, hence $\frac{[KI_BI_C]}{[JI_BI_C]} = \frac{KH}{HJ} = \frac{KQ}{QJ} = \frac{4+x}{x} = \boxed{\frac{103}{71}}$.

   Comment: Puhua Cheng contributed to the above solution. Here is the author's original roundabout proof of the claim that $(HQ; KJ) = -1$: Pascal on I_BEGI_CDF shows that the six points lie on a conic. The ellipse gives pairs of some involution $\psi : (HH), (JK), (DE)$. To see that (QQ) is also a pair, it suffices to show that $(HQ; DE) = -1$ but this follows from projecting the bundle from point of infinity on line AQ onto line I_BI_C and we know that $(HA; I_BI_C) = -1$.