CMIMD 2021

Combinatorics & CS Div. 2 Solutions

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1. We have a 9 by 9 chessboard with 9 kings (which can move to any of 8 adjacent squares) in the bottom row. What is the minimum number of moves, if two pieces cannot occupy the same square at the same time, to move all the kings into an X shape (a 5×5 region where there are 5 kings along each diagonal of the X, as shown below)?

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O       O
  O   O
    O
  O   O
O       O
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Proposed by David Tang

Answer: 18

Solution: The optimal X is centered and touching the bottom. Every king can only move up at most 1 tile per move. If we count the number of "up" moves necessary, we find that we need at least 18 up moves (with equality if the X is touching the bottom). It is easily achievable by just moving the pieces greedily, hence the answer is $\boxed{18}$.

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2. Dilhan has objects of 3 types, A, B, and C, and 6 functions

$$f_{A,B}, f_{A,C}, f_{B,A}, f_{B,C}, f_{C,A}, f_{C,B}$$

where $f_{X,Y}$ takes in an object of type X and outputs an object of type Y. Dilhan wants to compose his 6 functions, without repeats, such that the resulting expression is well-typed, meaning an object can be taken in by the first function, and the resulting output can then be taken in by the second function, and so on. In how many orders can he compose his 6 functions, satisfying this constraint?

Proposed by Adam Bertelli

Answer: 18

Solution: Imagine we write the three letters A, B, C at equally spaced points around a circle. Then, we can imagine each function as an edge taking one type to another, moving either clockwise or counterclockwise, which we will denote C and W, respectively. The possible sequences that allow us to traverse each edge once are simply the 6 cyclic shifts of the string CCCWWW, and since there are 3 possible starting locations A, B, C, the total number of sequences of edges we can obtain is $6 \cdot 3 = \boxed{18}$.

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3. Adam has a box with 15 pool balls in it, numbered from 1 to 15, and picks out 5 of them. He then sorts them in increasing order, takes the four differences between each pair of adjacent balls, and finds exactly two of these differences are equal to 1. How many selections of 5 balls could he have drawn from the box?

Proposed by Adam Bertelli

Answer: 990

Solution: There are two cases:

- If the two differences that are equal to 1 occur between disjoint pairs of elements (e.g. 1,2 and 4,5), then this is equivalent to selecting 3 objects out of 11, adding two dividers between pairs of objects to ensure that objects we do not want to be adjacent are not adjacent, and finally choosing two of our objects to represent a pair of adjacent balls instead of one ball. In total, this can occur in $\binom{11}{3}\binom{3}{2} = 465$ ways.
- If instead the two differences that are equal to 1 have a shared element (e.g. 1,2 and 2,3), then this is again equivalent to selecting 3 objects out of 11, adding two dividers, and choosing one of our objects to represent 3 adjacent balls, which gives a total of $\binom{11}{3}\binom{3}{1} = 465$ ways.

Thus in total there are $465 + 465 = \boxed{990}$ ways to select these 5 balls.

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4. Vijay has a stash of different size stones: in particular, he has 2021 types of stones, with sizes from 0 through 2020, and he has $2r + 1$ stones of size r.

   Vijay starts randomly (and without replacement) taking out stones from his stash and laying them out in a line. Vijay notices that the first stone of size 2020 comes before the first stone of size 2019, the first stone of size 2019 is before the first stone of size 2018, and so on. What is the probability of this happening?

   Express your answer in terms of only basic arithmetic operations (division, exponentiation, etc.) and the factorial function.

   Proposed by Misha Ivkov

   Answer: $\dfrac{4042!}{2^{2021}(2021!)^3}$

   Solution: The probability that the first stone is size 2020 is $\frac{4041}{2021^2}$. Then, we simply need to get a stone of size 2019 first out of all the stones of size $0, 1, \ldots, 2019$. So, this probability is $\frac{4039}{2020^2}$. we can continue this reasoning all the way down to get the probability as

$$\frac{4041!!}{(2021!)^2} = \boxed{\frac{4042!}{2^{2021}(2021!)^3}}.$$

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5. Bill Gates and Jeff Bezos are playing a game. Each turn, a coin is flipped, and if Bill and Jeff have $m, n > 0$ dollars, respectively, the winner of the coin toss will take $\min(m, n)$ from the loser. Given that Bill starts with 20 dollars and Jeff starts with 21 dollars, what is the probability that Bill ends up with all of the money?

   Proposed by Daniel Li

   Answer: $\dfrac{20}{41}$

   Solution: First, note that the game ends. On each turn, the probability that the game ends is $\frac{1}{2}$. Note $\lim_{n\to\infty} (\frac{1}{2})^n = 0$. The expected value of winnings for Bill on any turn is 0. Thus, for any turn n, Bill is expected to have 20 dollars, and so when the game ends, Bill is expected to have won $\boxed{\dfrac{20}{41}}$ of the time.

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6. Adam is playing Minesweeper on a 9×9 grid of squares, where exactly $\frac{1}{3}$ (or 27) of the squares are mines (generated uniformly at random over all such boards). Every time he clicks on a square, it is

either a mine, in which case he loses, or it shows a number saying how many of the (up to eight) adjacent squares are mines.

First, he clicks the square directly above the center square, which shows the number 4. Next, he clicks the square directly below the center square, which shows the number 1. What is the probability that the center square is a mine?

Proposed by Adam Bertelli

Answer: $\dfrac{88}{379}$

Solution: Note that the squares touching both the 1 and the 4 form a rectangle of 3 squares. We will consider two separate cases on this rectangle:

- If the central rectangle contains one mine, then there are $\binom{5}{3} = 10$ ways to place the mines around the 4, 3 ways to place the mine in the central rectangle, and $\binom{66}{23}$ ways to place the remaining 23 mines in squares that do not touch either the 1 or the 4. Note that in this case, there is a $\frac{1}{3}$ probability of the center square being a mine.
- If the central rectangle contains no mines, then there are $\binom{5}{4} = 5$ ways to place the mines around the 4, 5 ways to place the mine around the 1, and $\binom{66}{22}$ ways to place all of the remaining mines. Note that in this case, there is no chance the center square is a mine.

Thus our total probability for the center square being a mine is

$$\frac{\frac{1}{3}\cdot 3\cdot 10\cdot \binom{66}{23}}{3\cdot 10\cdot \binom{66}{23} + 5\cdot 5\cdot \binom{66}{22}} = \frac{2\cdot \binom{66}{23}}{6\cdot \binom{66}{23} + 5\cdot \frac{23}{44}\cdot \binom{66}{23}} = \frac{2}{6+\frac{115}{44}} = \boxed{\frac{88}{379}}.$$

Comment: This problem illustrates the more general fact that, for minesweeper boards with density < 0.5, individual configurations with less mines are relatively *more likely* to be correct, a useful trick often overlooked by minesweeper players.

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7. How many permutations of the string 0123456 are there such that no contiguous substrings of lengths $1 < \ell < 7$ have a sum of digits divisible by 7?

   Proposed by Srinivasan Sathiamurthy

   Answer: 420

   Solution: Add 0 at the end, multiplying the total answer by 5 (we can insert it anywhere but the ends).

   If we first require that no two adjacent digits sum to 7, there are 4 possible shapes of strings, where each letter denotes some pair from (1,6), (2,5), (3,4):

   ABCABC
   ABCACB
   ABCBAC
   ABACBC

   The number of ways to place digits in each shape without restrictions is $2^3 \cdot 3! = 48$. The only additional restrictions are the tuples (1,2,4) and (3,5,6), which we can count using PIE. There are 4, 3, 3, and 2 places where such a contiguous subset could occur in each of the 4 shapes respectively, and there are 3! ways to rearrange each instance, so in total $3! \cdot 12 = 72$ strings contain (1,2,4) in some order, and by symmetry 72 contain (3,5,6). The only way to contain both is to have the first half be one, and the second half be the other, which only works in the first 3 shapes, and within these we can swap the two halves and rearrange one of the two halves, giving $2 \cdot 3! = 12$ degrees of freedom, thus the final answer is $4 \cdot 48 - 2 \cdot 72 + 3 \cdot 12 = 84$, which gives $\boxed{420}$ when multiplied by 5.

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8. Suppose you have a 6 sided dice with 3 faces colored red, 2 faces colored blue, and 1 face colored green. You roll this dice 20 times and record the color that shows up on top. What is the expected value of the product of the number of red faces, blue faces, and green faces?

Proposed by Daniel Li

Answer: 190

Solution: Consider the set S of ordered triples (i, j, k) such that the ith roll is red, the jth roll is green, and the kth roll is blue. It is clear that the number of such triples is rgb, so we just have to find the expected value of the size of this set.

For each of the $20 \cdot 19 \cdot 18$ possible such triples (i, j, k), it will be a part of S with probability $\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{6}$. By Linearity of Expectation

$$\mathbb{E}[|S|] = \mathbb{E}\left[\sum_{(i,j,k)\in S} 1\right] = \sum_{(i,j,k)} \mathbb{P}[(i,j,k) \in S] = (20 \cdot 19 \cdot 18) \cdot \frac{1}{36} = \boxed{190}$$

Alternate Solution (without Linearity of Expectation): Let the number of red faces be r, blue faces be b, and green faces be g. We will compute for n dice rolls. The expected value is the sum of possible products weighted by the probability of that outcome. Thus, we compute

$$\sum_{r+b+g=n} (rbg) \left(\left(\frac{1}{2}\right)^r \left(\frac{1}{3}\right)^b \left(\frac{1}{6}\right)^g\right) \binom{n}{r, g, b}$$

We expand the multinomial and factor constants to get

$$= \sum_{r+b+g=n} (rbg) \left(\left(\frac{1}{2}\right)^r \left(\frac{1}{3}\right)^b \left(\frac{1}{6}\right)^g\right) \frac{n!}{r!g!b!}$$

$$= \sum_{r+b+g=n} \left(\left(\frac{1}{2}\right)^r \left(\frac{1}{3}\right)^b \left(\frac{1}{6}\right)^g\right) \frac{(n-3)!}{(r-1)!\,(g-1)!\,(b-1)!} n\,(n-1)\,(n-2)$$

$$= \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(\frac{1}{6}\right) \sum_{r+b+g=n} \left(\left(\frac{1}{2}\right)^{r-1} \left(\frac{1}{3}\right)^{b-1} \left(\frac{1}{6}\right)^{g-1}\right) \frac{(n-3)!}{(r-1)!\,(g-1)!\,(b-1)!} n\,(n-1)\,(n-2)$$

$$= \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(\frac{1}{6}\right) n\,(n-1)\,(n-2) \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{6}\right)^{n-3} = \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(\frac{1}{6}\right) n\,(n-1)\,(n-2)\,.$$

Thus, as we have 20 dice rolls, the desired expected value is $\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(\frac{1}{6}\right) \cdot 20 \cdot 19 \cdot 18 = \boxed{190}$.